Modeling a vibrating string terminated against a bridge with arbitrary geometry

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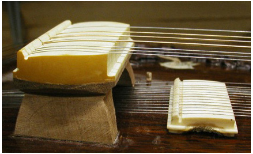
Motivation

In numerous musical instruments the collision of a vibrating string with rigid spatial obstacles, such as frets or a bridge is present.

Biwa

Shamisen

Sitar



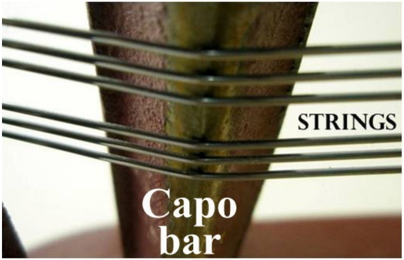
Jawari (bridge)

Motivation

Medieval and Renaissance bray harp and bray pins

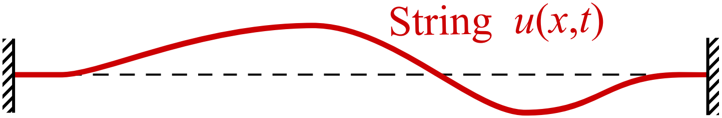
Audio example of bray harp timbre (15 s)

Motivation



Capo bar (Capo d'astro) of the piano cast iron frame

String description



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c = \sqrt{\frac{T}{\mu}} \tag{1}$$

$$u(0,t) = u(L,t) = 0 \tag{2}$$

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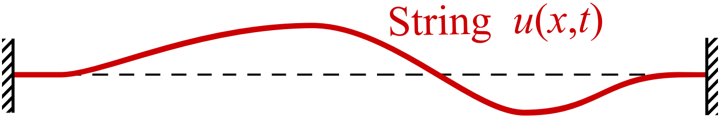

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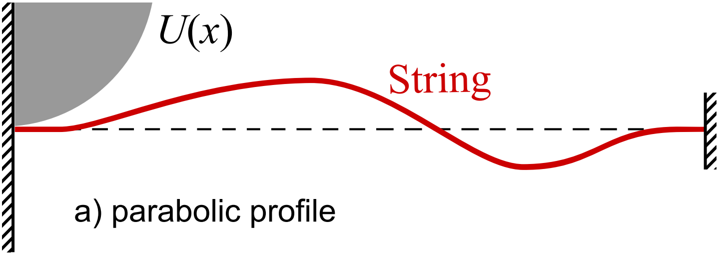
Solution to Eq. (1) is famous d'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left[u_r(x-ct) + u_l(x+ct)\right] \tag{3}$$

Geometric termination condition (TC)

TC is an absolutely rigid unilateral constraint of the string's transverse deflection.
Support profile geometry is described by an arbitrary function $U(x)$.



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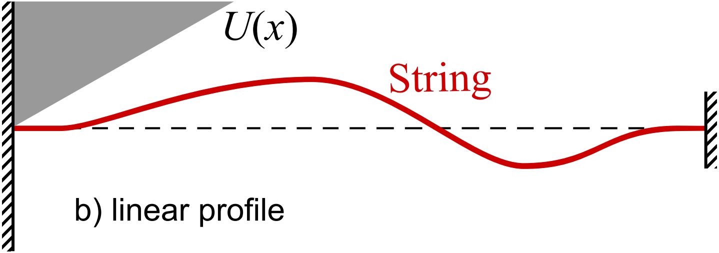

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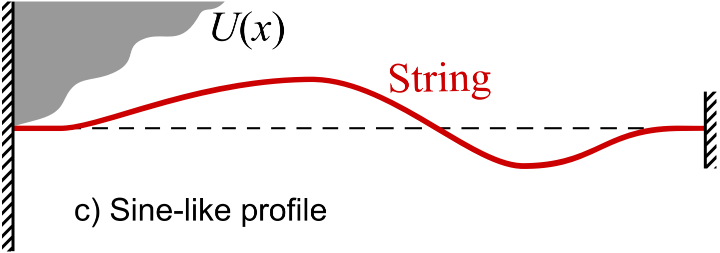

Bridge-string interaction model

Since the termination is rigid, it must hold

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In order to satisfy condition (4) for $u(x^*, t) > U(x^*)$ a reflected traveling wave is introduced

$$u_r\left(t - \frac{x^*}{c}\right) = U(x^*) - u_l\left(t + \frac{x^*}{c}\right), \tag{5}$$

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here the waves u_l and u_r correspond to any waves that have reflected from the terminator earlier.

Single wave reflection from the terminator

$$u_r(t - x^*/c) = U(x^*) - u_l(t + x^*/c)$$

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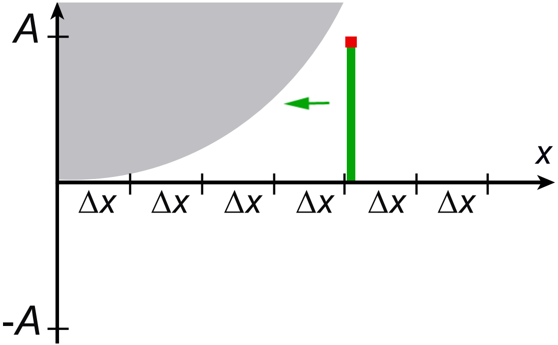

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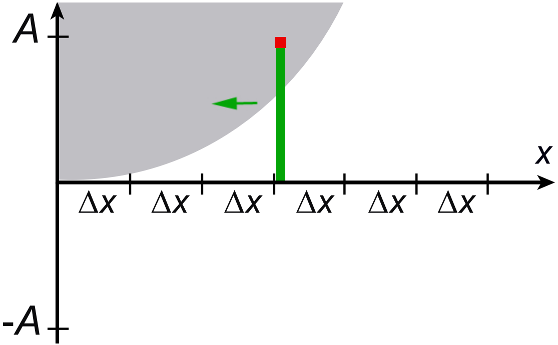

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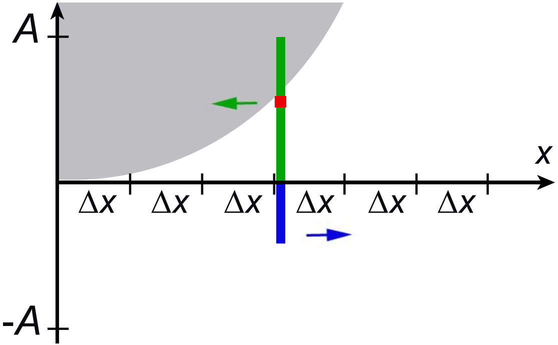
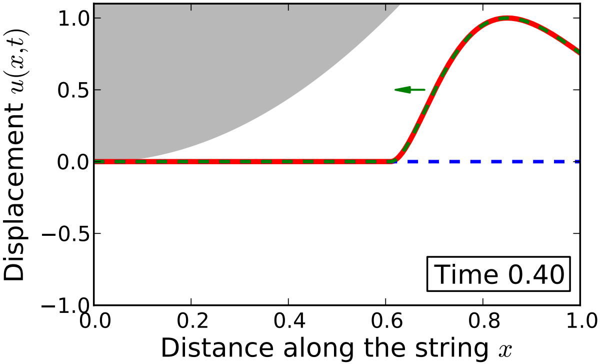


Single wave reflection from the terminator

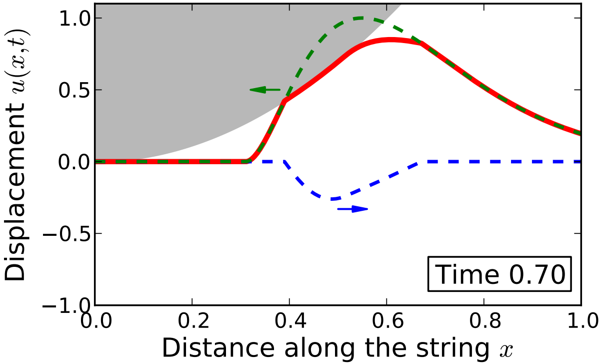
$$u_r(t - x^*/c) = U(x^*) - u_l(t + x^*/c)$$
$$u(x,t) = u_r(t - x/c) + u_l(t + x/c) \qquad (7)$$



Single wave reflection from the terminator

$$\color{blue}{u_r(t - x^*/c)} = U(x^*) - \color{green}{u_l(t + x^*/c)}$$

$$\color{red}{u(x,t)} = \color{blue}{u_r(t - x/c)} + \color{green}{u_l(t + x/c)} \tag{7}$$



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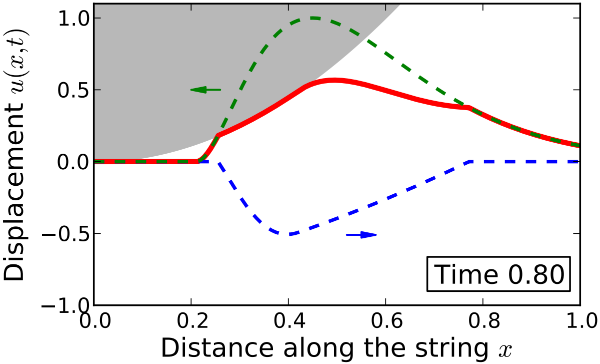

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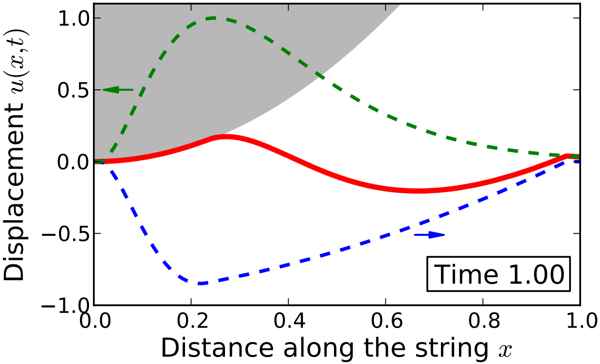
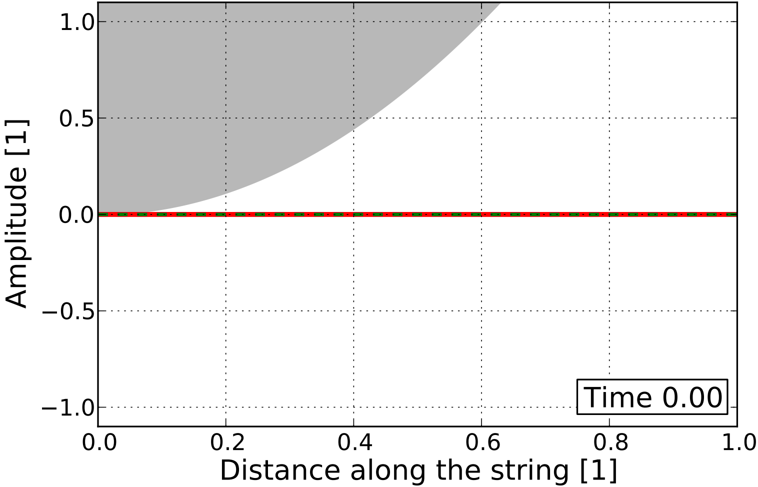


Single wave reflection from the terminator



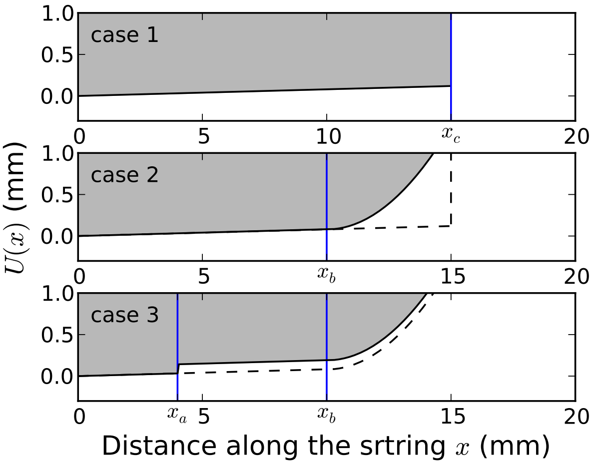
Model application: Biwa

String length $L = 0.8$ m
String plucking point $l = 3/4L = 0.6$ m
Linear mass density of the string $\mu = 0.375$ g/m
String tension $T = 38.4$ N
Velocity of the traveling waves $c = 320$ m/s
Fundamental frequency $f_0 = 200$ Hz

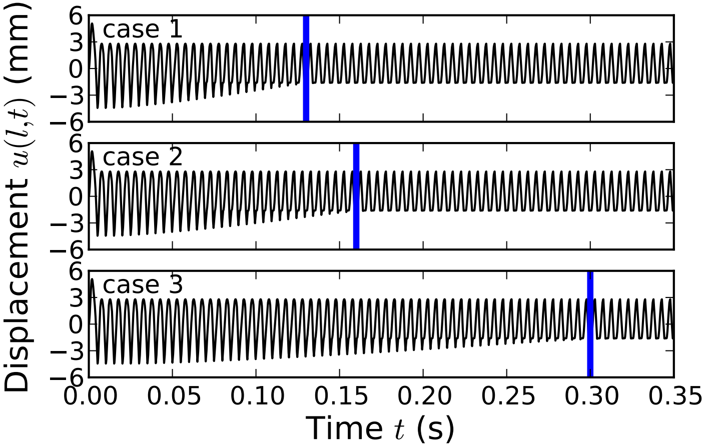
Bridge profiles studied

Profile shapes

- Case 1: Linear bridge with sharp edge
- Case 2: Linear bridge with curved parabolic edge
- Case 3: Bridge with minor defect



Result: Time series $u(l, t)$



Nonperiodic and *almost* periodic vibration regimes.

Case 1: Linear bridge with sharp edge

Spectrograms of the string vibration $u(l,t)$.

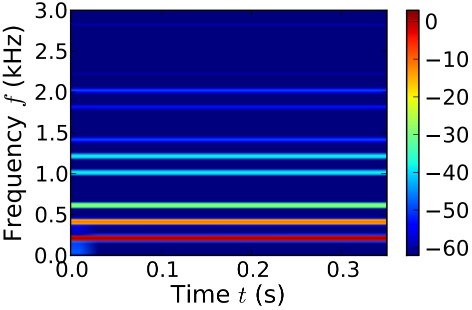


Figure: Linear case, no TC

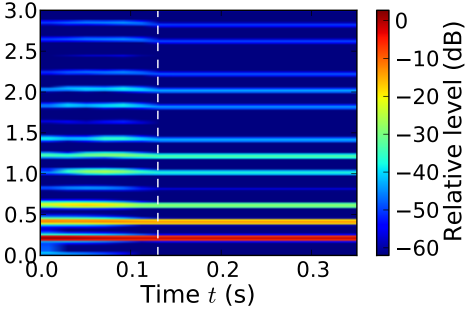
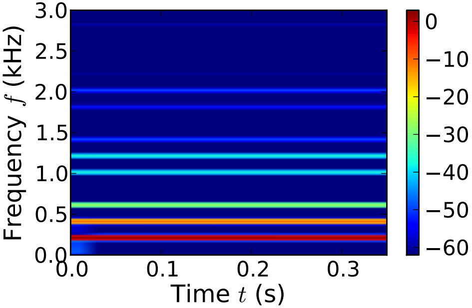


Figure: Case 1. Transition between the vibration regimes is shown by dashed line at $t_{\text{np}} = 0.13$ s.

Case 2: Linear bridge with curved edge

Spectrograms of the string vibration $u(l, t)$.



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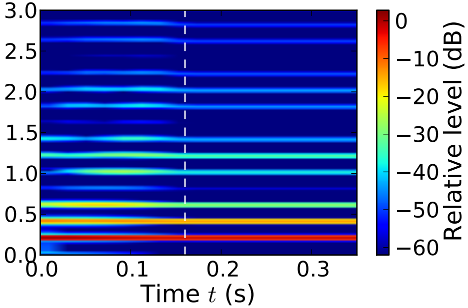
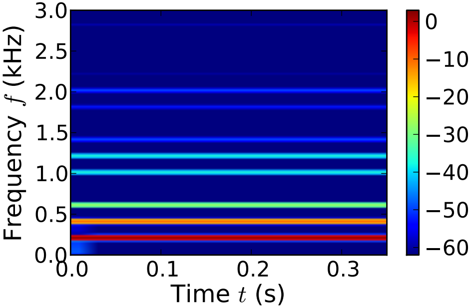


Figure: Case 2. Transition between the vibration regimes is shown by dashed line at $t_{\mathrm{np}} = 0.16$ s.

Case 3: Bridge with minor defect

Spectrograms of the string vibration $u(l,t)$.



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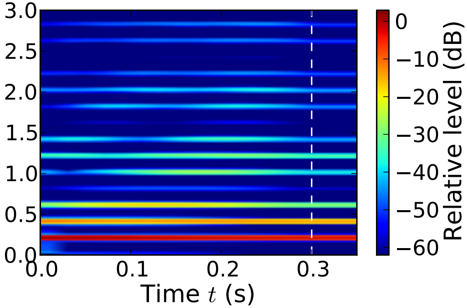
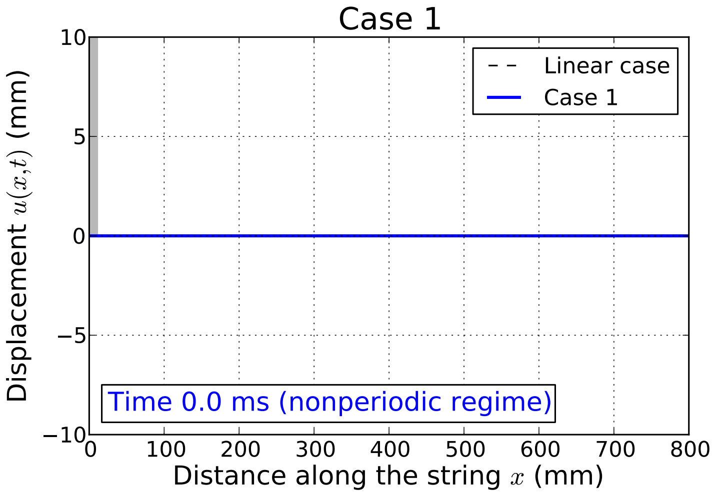


Figure: Case 3. Transition between the vibration regimes is shown by dashed line at $t_{\text{np}} = 0.3$ s.

Case 2: Animation



Conclusions

- A relatively simple method for modeling the TC-string interaction problem was presented.
- Two distinct vibration regimes in the case of the lossless string: strongly nonlinear nonperiodic and almost periodic regimes.
- Duration of the nonperiodic vibration regime depended on the bridge profile and on the plucking condition.
- A minor imperfection of the bridge profile geometry leads to prolonged nonperiodic vibration regime.