String Collision and Sliding Against a Smooth Obstacle in a Non-Planar Vibration Setting

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The problem

Simulation of string vibration against a stationary smooth barrier

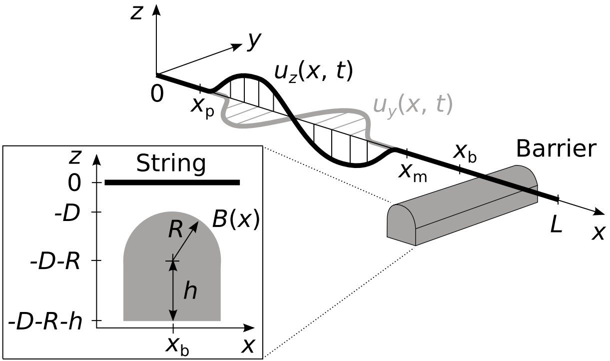

Figure: The problem schematic where $u_z(x,t)$ and $u_y(x,t)$ are the string displacements in the vertical and horizontal vibration planes, respectively. L is the string length, x_p is the plucking point, x_m is the measurement point and x_b is the barrier position.

Motivation and aim

Many stringed instruments are equipped with fretboards that strings can **collide** and **slide** against. Such collisions take place, for example, in guitars and various other lutes. The physics of this problem is **highly nonlinear** and multifaceted.

- We are interested in a **numerically robust** model.
- The **simplest** useful model.

String vibration

The problem is divided into **two coupled phenomena**:

1. The string–barrier collision in the vertical vibration plane.
2. The dry sliding friction in the horizontal vibration plane happening during the string–barrier contact.

The **ideal string vibration** in a single vibration plane is described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u(x,t)$ is the displacement, $c = \sqrt{T/\mu}$ is the speed of the waves travelling on the string, T is the tension and μ is the linear mass density of the string. Eq. (1) has an analytic solution referred to as the d'Alembert formula

$$u(x,t) = r(x - ct) + l(x + ct), \tag{2}$$

where r is the **travelling wave** propagating to the *right* and l is the **travelling wave** propagating to the *left*.

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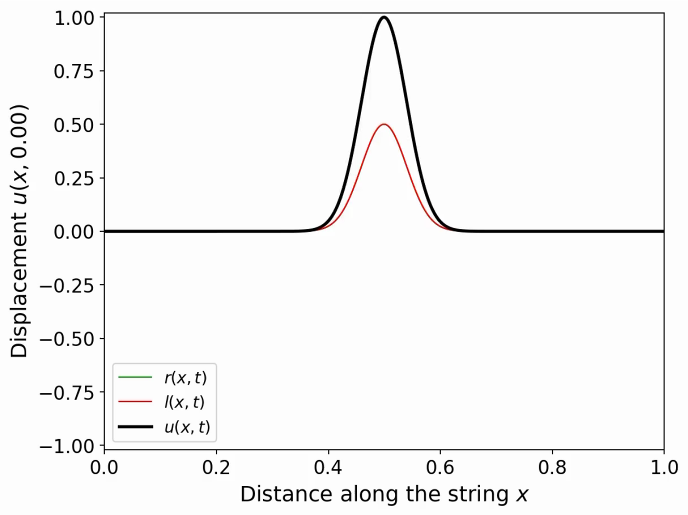
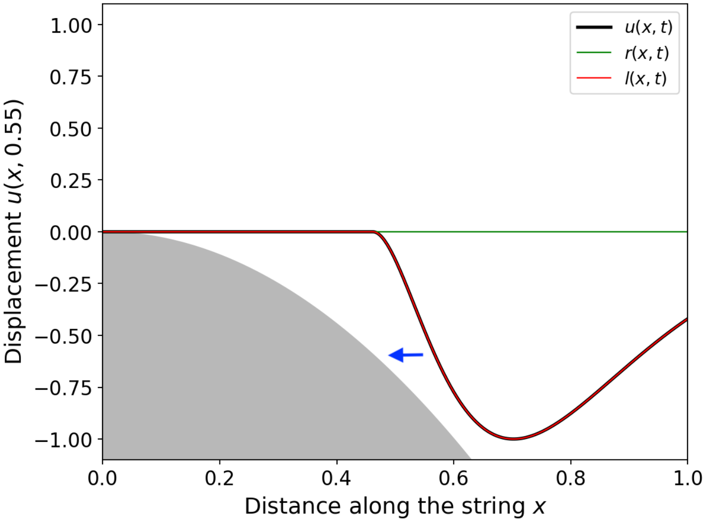
Travelling wave solution



In addition to (1) the boundary conditions $u(0,t) = u(1,t) = 0$ apply.

String–barrier collision model



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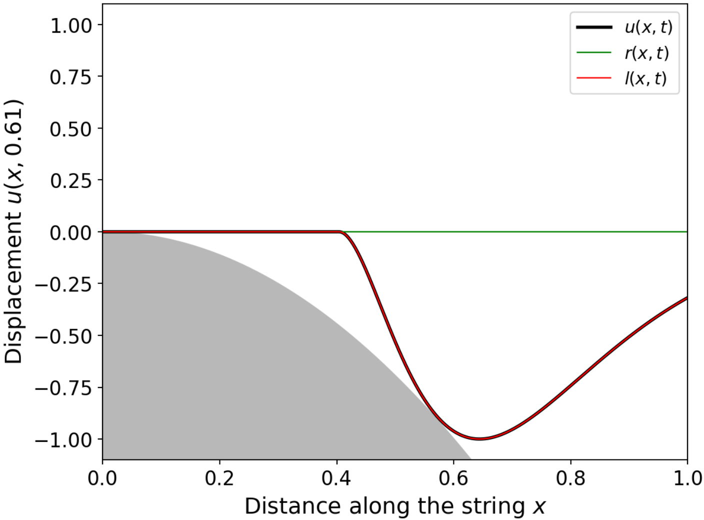

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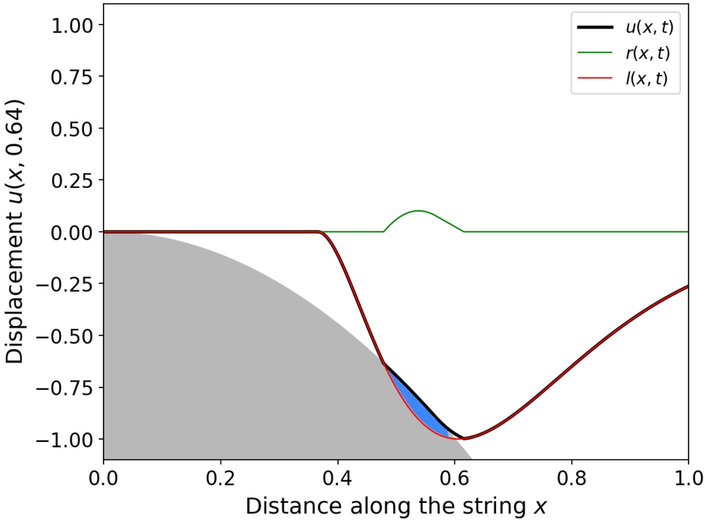
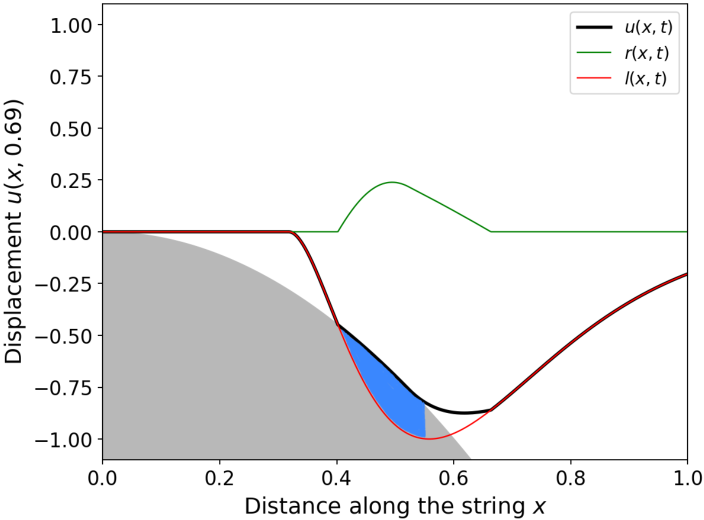


String–barrier collision model



String–barrier collision model

The above procedure can be written down as follows:

$$r\left(t-\frac{x^*}{c}\right)=\overbrace{B(x^*)-l\left(t+\frac{x^*}{c}\right)}^{\textit{virtual}\ \text{compression}\ z(x,t)}, \tag{3}$$

$$u(x^*,t)=B(x^*)=r\left(t-\frac{x^*}{c}\right)+l\left(t+\frac{x^*}{c}\right), \tag{4}$$

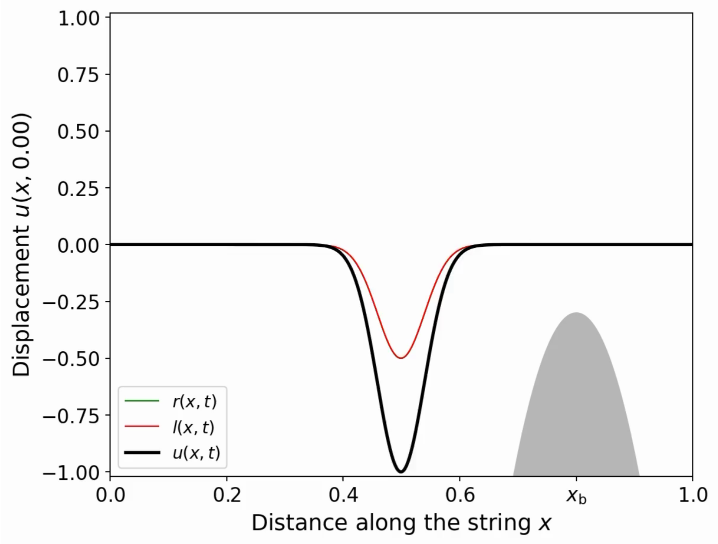
where

$$x^*|_t=\{x \mid 0\leq x\leq L\ \wedge\ B(x)>u(x,t)\}\,. \tag{5}$$

Coordinate x^* denotes the spatial point where the string *wants to* penetrate the barrier. Above, L is the string length, $B(x)$ is the cross-section profile of the barrier segment.

Waves approaching and colliding from the left side are reflected in a symmetrically opposite manner.

String–barrier collision model



Dry sliding friction model

Friction force F_f is defined as follows:

$$F_\mathrm{f} = \Psi(v_{y\mathrm{b}}) \cdot F, \qquad (6)$$

$$\Psi(v_{y\mathrm{b}}) = \mathrm{sgn}(v_{y\mathrm{b}}) \frac{\mathrm{e}^{-\beta |v_{y\mathrm{b}}|} + \psi}{1 + \psi}, \qquad (7)$$

$$F(t) = E \cdot C(t) \int_{C(t)} \frac{z(x,t)}{B_0(x)} \mathrm{d}x, \qquad (8)$$

where F is the string–barrier contact force in z-direction and $\Psi(v_{y\mathrm{b}})$ is the two-parameter friction coefficient that depends on sliding velocity $v_{y\mathrm{b}}$, z is the *virtual* compression, E is a Young's modulus type constant, C is the length of string–barrier contact line (projected to x-axis) and B_0 is the thickness of the barrier in z-direction.

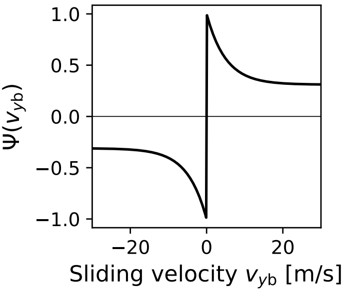


Figure: Friction characteristics curve — the dynamic friction coefficient where $\beta = 0.2$ s/m and $\psi = 0.45$.

The whole model

$$u_z(0,t) = u_y(0,t) = u_z(L,t) = u_y(L,t) = u_z(x,0) = u_y(x,0) = 0, \quad (9)$$

$$\frac{\partial^2 u_z}{\partial t^2} = c^2 \frac{\partial^2 u_z}{\partial x^2} - \underbrace{2\gamma \frac{\partial u_z}{\partial t}}_{\text{losses}} + \underbrace{\delta(x - x_\mathrm{p}) F_{z\mathrm{p}}}_{\text{string plucking}}, \quad \leftarrow \boxed{\text{String–barrier collision model}} \quad (10)$$

$$\frac{\partial^2 u_y}{\partial t^2} = c^2 \frac{\partial^2 u_y}{\partial x^2} - \overbrace{2\gamma \frac{\partial u_y}{\partial t}}^{\text{losses}} + \overbrace{\delta(x - x_\mathrm{p}) F_{y\mathrm{p}}}^{\text{string plucking}} - \overbrace{\delta(x - x_\mathrm{b}) \underbrace{\Psi(v_{y\mathrm{b}}) F}_{F_\mathrm{f}}}^{\text{dry sliding friction}}, \quad (11)$$

where

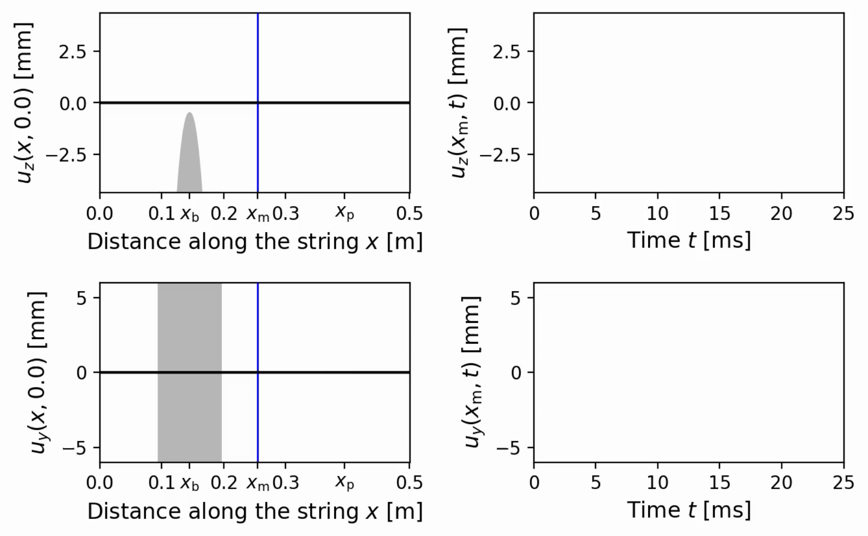
$$F(t) = E \cdot C(t) \int_{C(t)} \frac{z(x,t)}{B_0(x)} \mathrm{d}x, \quad (12)$$

$$\Psi(v_{y\mathrm{b}}) = \mathrm{sgn}(v_{y\mathrm{b}}) \frac{\mathrm{e}^{-\beta |v_{y\mathrm{b}}|} + \psi}{1 + \psi}, \qquad v_{y\mathrm{b}} = \frac{\partial u_y(x_\mathrm{b}, t)}{\partial t}, \quad (13)$$

$$F_{z\mathrm{p}}(t) = F_\mathrm{p}(t) \sin(\alpha) \quad \text{and} \quad F_{y\mathrm{p}}(t) = F_\mathrm{p}(t) \cos(\alpha). \quad (14)$$

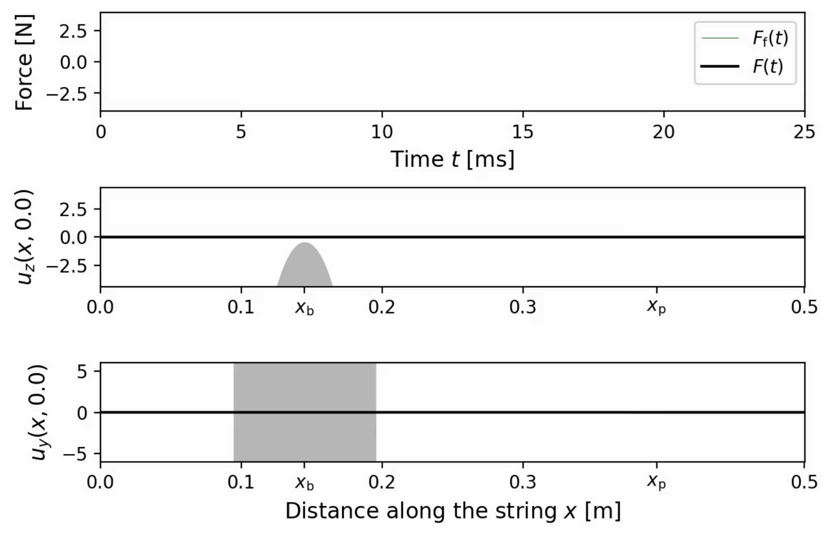
The second terms on the r.h.s. of (10) and (11) introduce the **frequency-independent loss** into the system.

Modelling results



Selected parameter values: $R = 50$ mm, $D = 0.5$ mm, $\beta = 0.2$ s/m, $\psi = 0.45$, $\gamma = 0$ 1/s, $f_0 = 196.96$ Hz ($T = 90$ N, $\mu = 2.32$ g/m), $\alpha = 0.2\pi$ rad.

Modelling results



Selected parameter values: See previous slide.

Modelling results

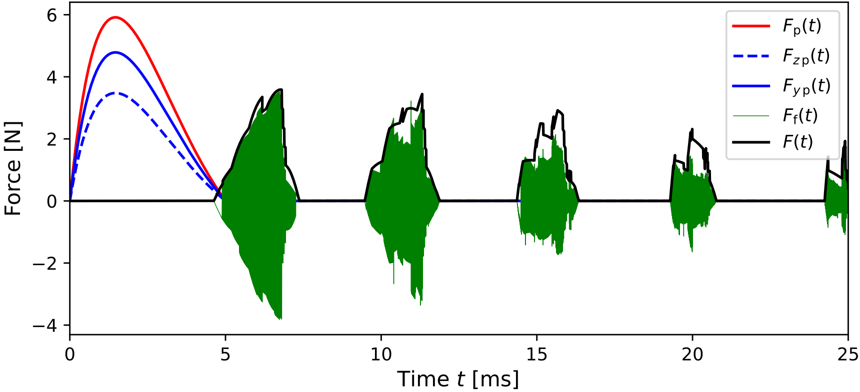


Figure: Force time-series related to the plucking condition, and the simulated contact/reactional and friction forces.

Selected parameter values: $x_{\mathrm{b}} = 5.5$ cm, $R = 50$ mm, $D = 0.5$ mm, $\beta = 0.2$ s/m, $\psi = 0.45$, $\gamma = 0$ 1/s, $f_0 = 196.96$ Hz ($T = 90$ N, $\mu = 2.32$ g/m), $\alpha = 0.2\pi$ rad.

Spectral analysis

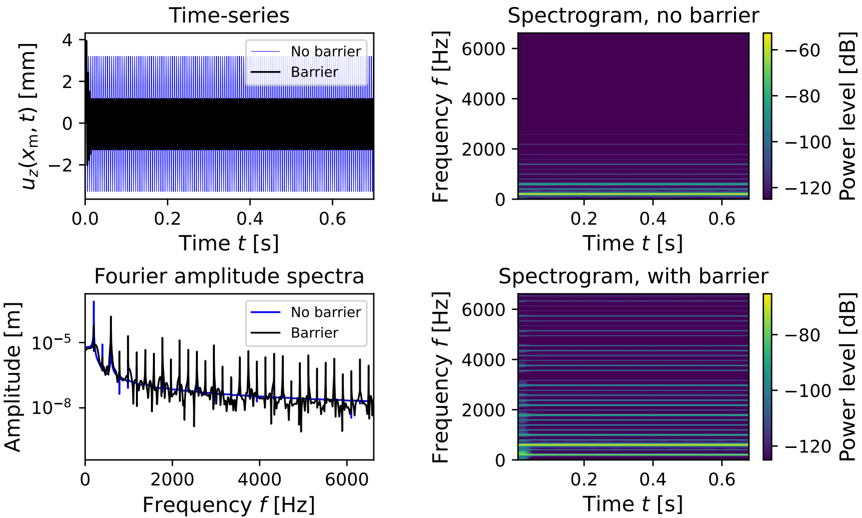


Figure: Comparison of obstructed and non-obstructed plucked string vibrations. The string is vibrating in the **vertical plane**. Parameter values shown on Slide 11.

Spectral analysis

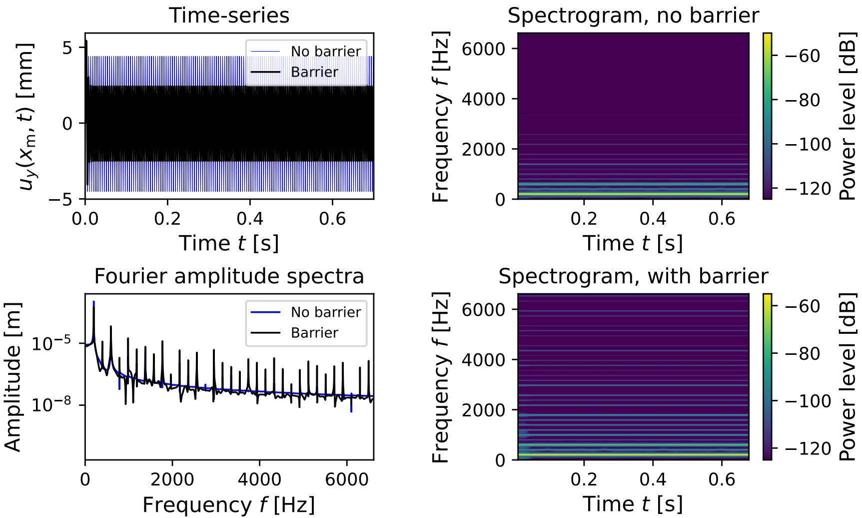


Figure: Comparison of obstructed and non-obstructed plucked string vibrations. The string is vibrating in the **horizontal plane**. Parameter values from Slide 11.

Spectral analysis

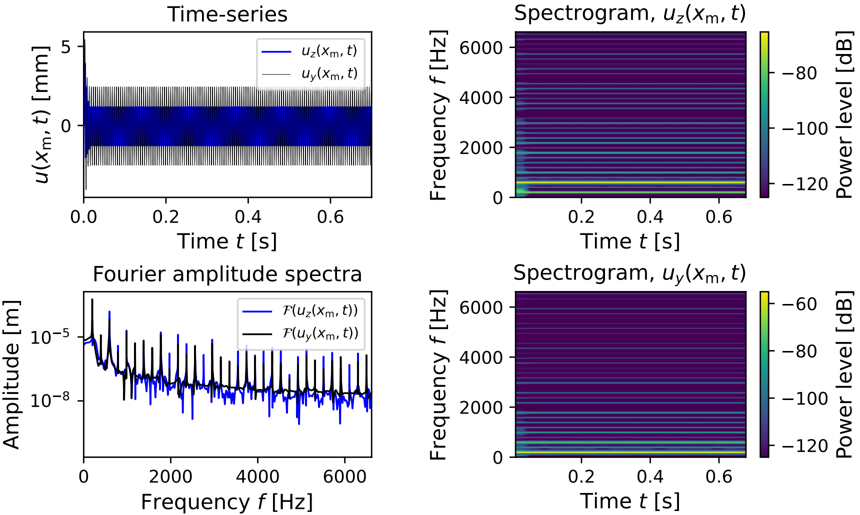


Figure: Comparison of vibrations taking place in the **vertical and horizontal** vibration planes. Parameter values shown on Slide 11.

Conclusions

- A novel **numerically robust time-stepping model** for simulating an elastic string vibration against a stationary obstacle in a dual-polarisation setting was presented.
- The string–obstacle collisions were modelled using a **purely kinematic** model.
- The dry sliding friction was modelled using a **physically sound model** that used as its input the *virtual* collision force history.
- The collision and friction models were coupled through the **virtual compression** of the barrier. We assumed a linearly elastic material that excepts forces only in a single direction.
- The simulated dynamics featured **two distinct vibration regimes**. The initial short-lasting regime characterised by the high energy collisions and the more peaceful regime characterised by the nonlinear inter-modal energy transfer.

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