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Nonlinear pulse propagation in microstructured materials in case of the negative group velocity

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Abstract. *Wave propagation in the Mindlin type microstructured material is studied. Here we follow the models from [1] derived for multiple micro-structures. It was shown in [2] that in case of hierarchical microstructure (a scale within scale) the negative group velocity (NGV) may arise. Dispersion analysis is used in order to determine the conditions for emergence of the NGV and numerical simulations of pulse propagation are carried out in order to demonstrate the effect of the NGV. The numerical solutions of approximated and full systems are compared.*

Keywords: Pseudospectral method; Boussinesq-type equations; Dispersion; Negative group velocity.

1 INTRODUCTION AND GOVERNING EQUATIONS

The driving force for the present study is to understand better the wave propagation in nonlinear materials with microstructures. For that purpose we apply dispersion analysis and take a look at the numerical solutions of the model equations describing a continuum with microstructure and nonlinearities in the macro– and microscales [1, 2]. The key aspect under consideration is how (and if) the existence of negative group velocity region in dispersion curve(s) affects the evolution of the solution. Secondary goal of the study is to assess the quality of approximated equation for double hierarchical microstructure [3] from the viewpoint of numerical simulation.
Briefly summarizing the main aspects of the governing equations under study. The physical background for the model equations is Mindlin-type microstructured environment (details can be found in [1, 4, 5] and references therein) with two distinct microstructures taken into account. The full system is considered against a hierarchical approximation derived in [3] which, in a nutshell, gets rid of the two higher order dispersion branches (so called optical branches) leaving it with only the lowest order dispersion branch (so called acoustical branch).
For governing equations we start with Lagrangian $L = K - W$, where K is kinetic and W is potential energy and get the governing equations by using Euler-Lagrange equations after determining the K and W. For two microstructures one of the simplest potentials W which accounts for nonlinear and dispersive terms can be taken as $W = \frac{Y}{2}u_x^2 + A_1\varphi_1 u_x + \frac{B_1}{2}\varphi_1^2 + \frac{C_1}{2}\varphi_{1x}^2 + A_{12}\varphi_{1x}\varphi_2 + \frac{B_2}{2}\varphi_2^2 + \frac{C_2}{2}\varphi_{2x}^2 + A_2\varphi_2 u_x + \frac{N}{6}u_x^3 + \frac{M_1}{6}\varphi_{1x}^3 + \frac{M_2}{6}\varphi_{2x}^3$, where u is macro displacement, φ_i are microdeformations and capital letters denote material coefficients. Subscript x denotes the spatial and t– the temporal partial derivative. If we take $A_{12} = 0$ then we get a double microstructure model where concurrent microstructures do not interact and taking $A_2 = 0$ results in a hierarchical microstructure model where the second microstructure is contained within the first one. In the following text we deal with the case $A_2 = 0$. Following Euler-Lagrange formalism the system of governing equations in the dimensionless form is

$$\begin{aligned}
&\frac{YU_0}{L^2}U_{TT} = \frac{YU_0}{L^2}U_{XX} + \frac{A_1 l_1}{L^2}\Phi_{1X} + \frac{NU_0^2}{L^3}U_X U_{XX} \\
&\frac{I_1}{\rho}\frac{Yl_1}{L^3}\Phi_{1TT} = \frac{C_1 l_1}{L^3}\Phi_{1XX} + \frac{A_{12} l_2}{L^2}\Phi_{2X} + \frac{M_1 l_1^2}{L^5}\Phi_{1X}\Phi_{1XX} - \frac{B_1 l_1}{L}\Phi_1 - \frac{A_1 U_0}{L}U_X \\
&\frac{I_2}{\rho}\frac{Yl_2}{L^3}\Phi_{2TT} = \frac{C_2 l_2}{L^3}\Phi_{2XX} - \frac{A_{12} l_1}{L^2}\Phi_{1X} + \frac{M_2 l_2^2}{L^5}\Phi_{2X}\Phi_{2XX} - \frac{B_2 l_2}{L}\Phi_2,
\end{aligned} \tag{1}$$

where ρ is density, I_i are microinertia, l_i are characteristic scales of the microstructures, U_0 is the amplitude and L is wavelength of the initial excitation. Capital U and Φ are used to emphasize that this is the dimensionless case. Applying slaving principle it is possible to derive an approximated equation from (1) [3]

$$u_{tt} = (c_0^2 - c_{A1}^2)u_{xx} + p_1^2 c_{A1}^2 \left[u_{tt} - (c_1^2 - c_{A2}^2)u_{xx}\right]_{xx} - p_1^2 c_{A1}^2 p_2^2 c_{A2}^2 \left(u_{tt} - c_2^2 u_{xx}\right)_{xxxx}. \quad (2)$$

Parameters c_i and c_{Ai} are related to velocities in the system and parameters p_i are time constants (detailed descriptions found in [3]). In essence the slaving principle gets rid of higher order dispersion branches (higher degrees of freedom) leaving only the acoustical dispersion branch. The advantages of Eq. (2) are discussed in detail in [3], however, here we focus on assessing the "quality" of the approximation from the numerical standpoint by comparing the solutions of the approximation (2) and full system (1) under selection of parameter combinations where NGV exists. It should be added that the equations are of the Boussinesq-type, i.e., these are second order equations (in terms of standard wave equation) with some kind of nonlinear and dispersive terms.

1.1 Numerical method, initial and boundary conditions

The pseudospectral method (PSM) is used: spatial derivatives are found using the Discrete Fourier Transform (DFT) and then the model equations are rewritten as a system of first order differential equations with respect to time which are solved with a numerical ODE solver. It should be noted that for applying the PSM the equation needs to be in a suitable form with only time derivatives on the right hand side of the equation and only spatial derivatives on the left hand side of the equation which is not a problem for (1) but is clearly not the case with Eq. (2). For solving Eq. (2) a change of variables is used for transforming it into a suitable format for direct PSM application. The change of variables is $H = u - \kappa_1 u_{xx} + \kappa_2 u_{xxxx}$, where $\kappa_1 = p_1^2 c_{A1}^2$ and $\kappa_2 = p_1^2 c_{A1}^2 p_2^2 c_{A2}^2$ This allows one to write Eq. (2) so that on the left hand side there is only term H_{tt} making it possible to apply the PSM directly. The final result is transformed back into terms of u for the purposes of comparing the solutions.
Periodic boundary conditions are used $U(X,T) = U(X + 2km\pi, T),\ m = 1, 2, \ldots,$ where $k = 128$ is length of the spatial period in terms of 2π sections. The initial condition is a bell shaped pulse described by sech^2 function. $U(X,0) = U_o \operatorname{sech}^2(B_o X),\ B_o = 1/8,\ U_o = 2,$ where U_o is initial amplitude and B_o is parameter related to the width of the initial pulse. In addition some initial conditions are needed for the microstructures for which the microstructures are assumed to be at rest initially so that $\varphi_1(X,0) = \varphi_2(X,0) = 0$ and $\varphi_1(X,0)_T = \varphi_2(X,0)_T = 0$.
The detailed list of references for this particular implementation of the numerical scheme can be found in [6] including a brief view on the accuracy and computational cost of the used scheme with Boussinesq-type equations in Appendix A.

2 DISPERSION ANALYSIS AND NUMERICAL RESULTS

We use combined parameters (which are related to the wave speeds in the system) $\gamma_{A_1} = \sqrt{(A_1^2\rho)/(B_1 Y)},\ \gamma_{A_{12}} = \sqrt{(A_{12}^2\rho)/(B_1 I_1 Y)},\ \gamma_1 = \sqrt{(C_1\rho)/(I_1 Y)},\ \gamma_2 = \sqrt{(C_2\rho)/(I_2 Y)}$. Parameters related to non-linearity are used separately. We use potential energy parameters A_1, A_{12}, C_1 and C_2 to control values of combined parameters. The rest of the material parameters are fixed at values $Y = 1, I_1 = 0.8, I_2 = 0.9, \rho = 2, B_1 = 0.5, B_2 = 0.7, A_2 = 0, M_1 = M_2 = N = 0.01$.
Under some parameter combinations dispersion curves can have a shape where there exists a negative group velocity (NGV) region for a certain range of wavelengths. Out of set of about 6500 dispersion curves (combined parameter step 0.1 between 0 and 1) 790 can be observed to have a NGV region in at least one of the dispersion branches for Eq. (1). We investigate a subset of 27 characteristic cases from the larger NGV set focusing on cases where the NGV region is present in the acoustic (lowest) dispersion branch and out of these 27 cases 4 are presented as detailed examples in the present paper. Four detailed examples shown are chosen to represent different possible (characteristic) NGV configurations for the hierarchical double microstructure model. Dispersion analysis for Eq. (1) can be found, for example, in [2] and for Eq. (2) in [3, 7].
Parameter sets:
(a) $\gamma_{A_1} = 0.1,\ \gamma_{A_{12}} = 0.5,\ \gamma_1 = 0.1,\ \gamma_2 = 0.9$; NGV for middle opt. and aco. branches. No NGV for approx.
(b) $\gamma_{A_1} = 0.3,\ \gamma_{A_{12}} = 0.8,\ \gamma_1 = 0.4,\ \gamma_2 = 0.9$; NGV for middle opt. and aco. branches. No NGV for approx.
(c) $\gamma_{A_1} = 0.4,\ \gamma_{A_{12}} = 0.6,\ \gamma_1 = 0.3,\ \gamma_2 = 0.7$; NGV for middle opt. and aco. branches. No NGV for approx.
(d) $\gamma_{A_1} = 0.7,\ \gamma_{A_{12}} = 0.3,\ \gamma_1 = 0.2,\ \gamma_2 = 0.3$. NGV for only aco. branch. Small NGV for approx.

2.1 Results

Typical solution of Eq. (1) can be seen in Figure 1, top left panel.

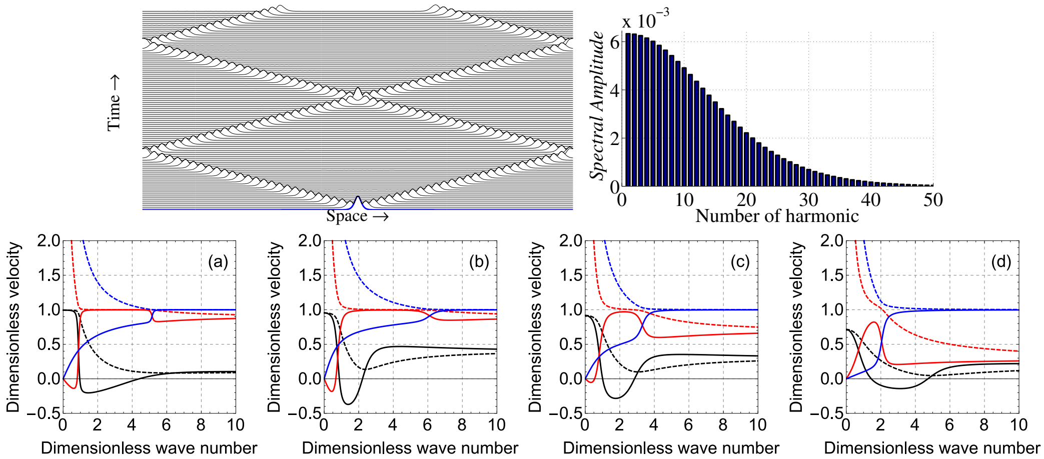

Figure 1: Top: A typical solution of Eq. (1). Time-slice plot of the solution (left), first 50 spectral amplitudes of the initial condition (right). Bottom: Group and phase speed plots for the parameter sets a,b,c,d — dashed lines – phase speeds, solid lines – group speeds, black – acoustic branch, red – middle optical branch, blue – top optical branch.

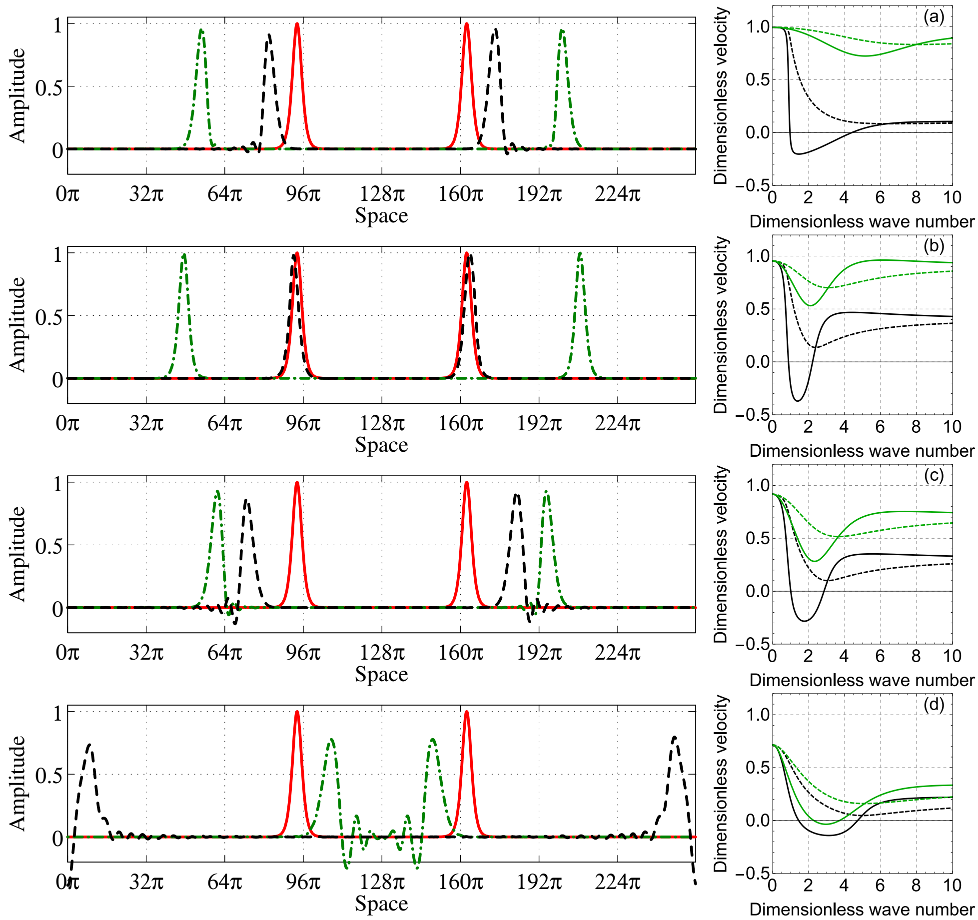

Figure 2: Solutions at $T = 1500$ for the parameter sets a,b,c,d (left) and comparison of the acoustical dispersion branches of the full system and approximation (right). Left panel: red solid line – wave equation, black dashed line – full system, green dash-dotted line – approximation. Right panel: black solid line – full system group speed, black dashed line – full system phase speed, green solid line – approximation group speed, green dashed line – approximation phase speed.

In principle the idea for detecting the effects of the NGV on the evolution of the solution is that some harmonic components of the solution are in the NGV region and as such it might be possible to pick up the effect in the solutions. The dispersion curves for Eq. (1) can be found in Figure 1 bottom panel. The optical branches are omitted from the phase– and group speed plots in Figure 2 for the sake of clarity where only the speeds from the acoustical branches of the full system (1) and approximation (2) are compared (right panels). Next, in Figure 2 snapshots of the solution waveprofiles at $T = 1500$ are presented for the full system and approximation as well as for the classical wave equation for reference (left panels).

Taking a look at Figure 2 one can note that for the parameter sets:

(a) – the shape and amplitude of the solutions of the full system and approximation are similar, however, the wave propagation speed is different; in the dispersion curves NGV regions exist for the middle optical and acoustical branches. The NGV region for acoustical branch is wide. There is no NGV region for the approximation;

(b) – similarly to the parameter set (a) the shape and amplitude of the full system and approximation solutions are similar and the wave propagation speeds are different; the match between dispersion curves is better but the NGV region for the approximation is still absent;

(c) – similar to the cases (a) and (b) the amplitudes of the solutions of the full system and approximation match well but there is significant difference between the propagation speeds of the pulses; both solutions have an oscillatory tail behind the pulse but the amplitude of the tail is smaller for the approximation; the match between the dispersion curves of the full system and approximation is better than in the cases of (a) and (b) but still not close enough to be considered a good match;

(d) – like in the cases (a), (b) and (c) the amplitude of the solutions matches well, however, although the dispersion curves match well to the point of approximation even having a NGV region there is still a significant gap between the propagation speeds of the solutions.

3 CONCLUSIONS

In summary following main conclusions can be drawn from the present study:

(i) Qualitatively the approximation is good as the shape and amplitude of the waveprofile are well represented.

(ii) Dispersion curves for the approximation can have the NGV region and approximation dispersion curve shapes are similar to the acoustic branches of the full system, however, for some parameter combinations the propagation speed of the approximated solution can be much different from the one in full system of equations.

(iii) The wave propagation speed of the approximation is slower than in the solution of the full system.

(iv) For the wide spectrum of frequencies (Figure 1) only a few are influenced by the NGV phenomenon. This is the reason why the existence of NGV has no significant effect on the evolution of pulse–type solutions under considered parameter combinations, initial condition and integration intervals.

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REFERENCES

[1] A. Berezovski, J. Engelbrecht, and T. Peets. Multiscale modeling of microstructured solids. *Mech. Res. Commun.*, 37, 531–534, 2010.

[2] T. Peets, D. Kartofelev, K. Tamm, and J. Engelbrecht. Waves in microstructured solids and negative group velocity. *EPL (Europhysics Lett.*, 103, 16001, 2013.

[3] J. Engelbrecht, F. Pastrone, M. Braun, and A. Berezovski. Hierarchies of waves in nonclassical materials. In P.P. Delsanto, editor, *Universality of Nonclassical Nonlinearity*, pp. 29–47, New York, 2006. Springer.

[4] J. Engelbrecht, A. Berezovski, F. Pastrone, and M. Braun. Waves in microstructured materials and dispersion. *Philos. Mag.*, 85, 4127–4141, 2005.

[5] A. Berezovski, J. Engelbrecht, and M. Berezovski. Waves in microstructured solids: a unified viewpoint of modeling. *Acta Mech.*, 220, 349–363, 2011.

[6] K. Tamm. *Wave Propagation and Interaction in Mindlin-Type Microstructured Solids: Numerical Simulation.* PhD thesis, Tallinn University of Technology, 2011.

[7] A. Berezovski, J. Engelbrecht, A. Salupere, K. Tamm, T. Peets, and M. Berezovski. Dispersive waves in microstructured solids. *Int. J. Solids Struct.*, 50, 1981–1990, 2013.

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