# Waves on a String of a Monochord Equipped with a Rigid Obstacle

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**Abstract**

This study presents a model capable of modelling wave propagation on a string of a monochord equipped with a rigid obstacle. The ideal string is allowed to vibrate in two mutually perpendicular planes. The obstacle spans both in the direction of the string at rest and perpendicular to it. String collision against the obstacle is modelled using a kinematic model that constrains the string displacement unilaterally. The string sliding against the obstacle is modelled using dry sliding friction model. The friction force is calculated using the string shape. The two vibration planes are coupled through the friction force. Modelling results show that the model generates a variety of dynamics, all strongly dependent on the initial conditions and system parameters. Interactions of a vibrating string with spatially distributed barriers, such as fretboards, play a significant role in the acoustics of many stringed instruments [1]. The proposed model can be applied to physics-based sound synthesis of these musical instruments.

**Keywords:** Ideal string, string–obstacle interaction, dry sliding friction, string instruments, numerical modelling.

## 1 Introduction

The wave equation and its general solution the d'Alembert formula are still finding innovative applications in acoustics, mechanics, and elsewhere *cf.* [2]. In this study a non-planar string vibration model that relies on the travelling wave solution is proposed. The ideal string is allowed to collide and slide against a rigid obstacle making the problem nonlinear, see Fig. 1.

## 2 Model

The ideal string vibration in $\mathbb{R}^3$ is divided into its vertical and horizontal components as shown in Fig. 1. Vertical string displacement $u_z(x,t)$ takes place in $zx$-plane and horizontal string displacement $u_y(x,t)$ in $yx$-plane. The string–obstacle interaction problem is also divided into

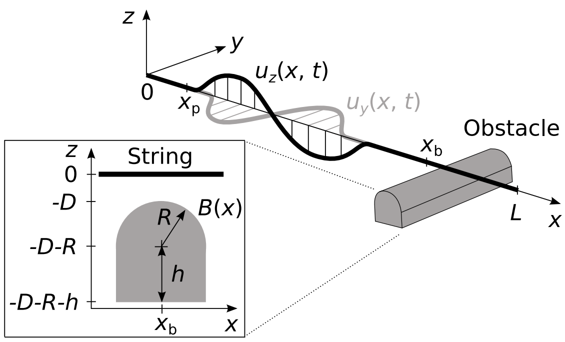


Figure 1: Schematic of the monochord. String length is $L$. String is excited at $x = x_\mathrm{p}$. Rigid obstacle with circular cross-section is places at the distance $D$ from the string at its rest and at $x = x_\mathrm{b}$ along the string extent.

two coupled phenomena: (i) String–obstacle collisions in the vertical vibration plane; (ii) Dry sliding friction in the horizontal vibration plane. The governing wave equations are the following:

$$\mathfrak{B}\left[\frac{\partial^2 u_z}{\partial t^2} = c^2\frac{\partial^2 u_z}{\partial x^2} - 2\gamma\frac{\partial u_z}{\partial t} + \delta(x - x_\mathrm{p})F_{z\mathrm{p}}\right], \quad (1)$$

$$\frac{\partial^2 u_y}{\partial t^2} = c^2\frac{\partial^2 u_y}{\partial x^2} - 2\gamma\frac{\partial u_y}{\partial t} + \delta(x - x_\mathrm{p})F_{y\mathrm{p}} - \delta(x - x_\mathrm{b})F_\mathrm{f}. \quad (2)$$

The initial and boundary conditions for both equations are the following:

$$u_z(0,t) = u_z(L,t) = u_z(x,0) = \frac{\partial u_z(x,0)}{\partial t} = 0, \quad (3)$$

$$u_y(0,t) = u_y(L,t) = u_y(x,0) = \frac{\partial u_y(x,0)}{\partial t} = 0. \quad (4)$$

String is excited by *plucking* it at $x = x_\mathrm{p}$, see the third terms on r.h.s. of the equations. External plucking force $F_\mathrm{p}$ components for both vibration planes are:

$$F_{z\mathrm{p}}(t) = F_\mathrm{p}(t)\sin(\alpha), \quad (5)$$

$$F_{y\mathrm{p}}(t) = F_\mathrm{p}(t)\cos(\alpha), \quad (6)$$

where $\alpha$ is the plucking angle measured with respect to $y$-axis and in $yz$-plane, $F_\mathrm{p}(t)$ is the

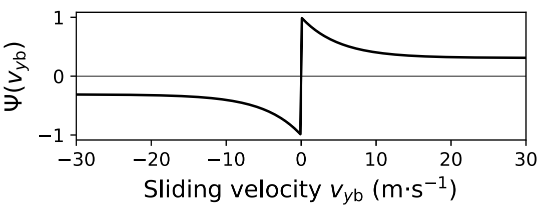


Figure 2: Friction characteristics curve where $\beta = 0.20\ \mathrm{s{\cdot}m^{-1}}$ and $\psi = 0.45$.

plucking force time-series. In the above wave equations $c = \sqrt{T/\mu}$ is the speed of travelling waves, $T$ is the tension, and $\mu$ is the linear mass density of the string. The second terms on r.h.s. of the equations introduce frequency-independent loss where $\gamma$ is the loss coefficient. In (1) $\mathfrak{B}$ denotes the calculations performed to model the string-obstacle collisions. The collision model is explained in Sec. 2.1. The last term on r.h.s. of (2) introduces friction force $F_f$ acting while the string is in contact with and sliding against the obstacle. The friction force calculation is explained in Sec. 2.2. Numeric integration of the model is not explained here.

### 2.1 String–obstacle collision model

In order to describe $u_z(x,t)$ during collisions a purely kinematic approach in agreement with the travelling wave solution is used. The solution to (1) without the loss and *plucking* terms, and for an infinite string has the following form:

$$u_z(x,t) = r_z(x - ct) + l_z(x + ct), \tag{7}$$

where $r$ is the travelling wave propagating along the string to the right, and $l$ is the travelling wave propagating to the left.

The proposed approach written down for travelling wave $r_z$ reflecting from the obstacle is the following:

$$r_z\Big(t - \frac{x'}{c}\Big) = B(x') - l_z\Big(t + \frac{x'}{c}\Big), \tag{8}$$

where $B(x)$ is the cross-section shape of the obstacle shown in Fig. 1, and where

$$x'|_t = \{x \,|\, x_\mathrm{b} \le x \le x_\mathrm{b} + R \,\wedge\, u_z(x,t) < B(x)\} \tag{9}$$

is the spatial point where $u_z$ would have penetrated the obstacle $B(x)$ in the absence of travelling wave (8). Since, (7) holds the resulting string displacement

$$u_z(x',t) = r_z\Big(t - \frac{x'}{c}\Big) + l_z\Big(t + \frac{x'}{c}\Big). \tag{10}$$

Reflected travelling wave $l_z$ is found similarly:

$$l_z\Big(t + \frac{x''}{c}\Big) = B(x'') - r_z\Big(t - \frac{x''}{c}\Big), \tag{11}$$

$$x''|_t = \{x \,|\, x_\mathrm{b} - R \le x < x_\mathrm{b} \,\wedge\, u_z(x,t) < B(x)\}. \tag{12}$$

### 2.2 Dry sliding friction model

Friction force $F_\mathrm{f}$ acting in the direction of $y$-axis and at $x = x_\mathrm{b}$ has the following form:

$$F_\mathrm{f} = \Psi(v_{y\mathrm{b}})F, \tag{13}$$

where the two-parameter friction characteristics

$$\Psi(v_{y\mathrm{b}}) = \mathrm{sgn}(v_{y\mathrm{b}})\frac{\mathrm{e}^{-\beta|v_{y\mathrm{b}}|} + \psi}{1 + \psi} \tag{14}$$

depends on the sliding velocity of the string $v_{y\mathrm{b}} = \partial u_y(x_\mathrm{b}, t)/\partial t$. Figure 2 shows an example of this characteristics curve. Normal force $F$ present in (13) is the string–obstacle contact force acting in the positive direction of $z$-axis and it has the form:

$$F(t) = -T\Bigg(\int_{x'|_t} \frac{\partial^2 r_z(x',t)}{\partial x^2}\mathrm{d}x + \int_{x''|_t} \frac{\partial^2 l_z(x'',t)}{\partial x^2}\mathrm{d}x\Bigg). \tag{15}$$

## 3 Results and conclusions

Modelling results feature two distinct vibration regimes. An initial short-lasting regime characterised by high energy collisions and a peaceful regime characterised by nonlinear inter-modal energy transfer, *cf.* [3].

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