INFLUENCE OF THE EDGE OF THE CAST IRON FRAME CURVATURE ON THE SPECTRUM OF THE PIANO STRING VIBRATION

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ABSTRACT

We investigate the vibrations of the ideal flexible string, which one end is rigidly clamped, and another one is terminated on the curved contact surface. The vibrating string repeatedly touches the termination and in turn, causes the modulation of fundamental frequency of the string, and the train of high frequency oscillations is generated. The problem is studied both analytically, and numerically. The effect of the contact nonlinearity and of the shape of the contact surface on of the spectral structure of the string vibration is considered. The influence of the impact amplitude on the vibration spectra of struck string is discussed.

1. INTRODUCTION

Investigation of the boundary condition of vibrating string is a very important problem in musical acoustics. It is well known that the fundamental frequency of piano string is strictly determined by the type of the string termination. The types of the string support in the piano are different for the bass and treble notes. All the far ends of the piano strings are terminated on the bass and treble bridges, which are the rather complicated resonant systems. The nearest ends of the bass and long treble strings begin from the agraffe that can be considered as an absolutely rigid clamp termination. But the most part of the treble strings starts from the edge of the cast iron frame. These strings turn the rigid edge over, and the tone of the strings vibration depends on the curvature of this termination. The similar type of the string support can be seen on the guitar and some other musical string instruments.

Usually the changing of tone caused by the curvature of the string support is negligible, but there is a family of Japanese plucked stringed instruments (biwa and shamisen), which sounding is strictly determined by the string termination [1, 2]. These lutes are equipped with a mechanism called "sawari" (touch). The sawari is a contact surface of very limited size, located at the nut-side end of the string, to which the string touches repeatedly, producing a unique timbre of the instrumental tone called the sawari tone.

This paper studies the influence of the geometrical nonlinearity of the string termination on the spectrum of its vibration.

2. SAWARI MODEL

The nonlinear model of sawari mechanism is considered in [3, 4] and the scheme of this model is shown in Figure 1.

It is assumed that the displacement $y(x,t)$ of the ideal (flexible) string of length L obeys the second-order wave equation. The right-hand end of the string is supported by the bridge, which is considered as a resonator. The left-hand end ($x = 0$) terminates at sawari surface, which is assumed to be rigid enough, and that is defined by $y = f(x)$. When the string pushes this surface from above, the sawari surface deforms, and, as a result, the repelling net force acts on a string.

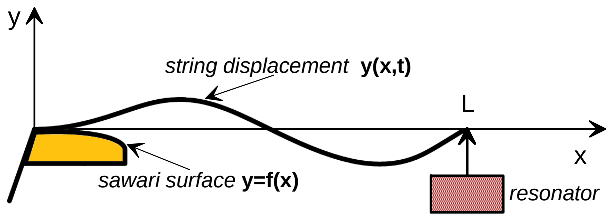

Figure 1: Scheme of sawari model.

The effect of sawari mechanism was studied also experimentally and numerically in [3, 4] . It was shown that the sawari excites a local disturbance of the string motion, which gets rich spectral components up to very large numbers of the fundamental frequency of the corresponding monochord (without sawari). It is evident, that similar mechanism of the contact nonlinearity can also generate the high frequency oscillations of the piano strings. In the following section will be presented another approach to the problem of vibration of the piano string with a nonlinear support.

3. PIANO STRING MODEL

The scheme of position of the treble piano string is shown in Figure 2. The left-hand end of the string wire is fastened to the cast iron frame. Then the string bends around the rigid edge of the frame, thereafter it runs over the piano bridge, and terminates again on the frame. The string is assumed to be ideal (flexible). The piano hammer strikes the string at the contact point $x = l$, and this generates two simple nondispersive traveling waves $y(t+x/c)$ and $y(t-x/c)$ moving in both directions. At the first moment the amplitude of these waves is always positive.

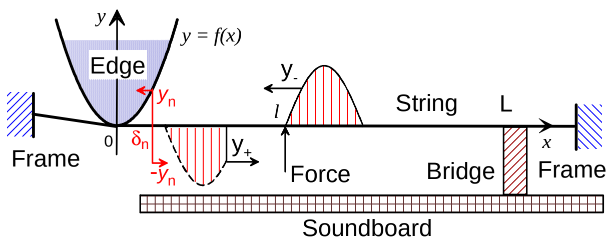

Figure 2: Scheme of piano string model.

Let's consider the wave $y_- = y(t + x/c)$ moving to the edge termination, which form is defined by the function $f(x)$. This wave reflects back from the edge, and the reflection occurs in such a manner that on the rigid surface at any moment t we have $y_- + y_+ = 0$. Thus, the amplitude of the reflected wave $y_+ = -y_-$. But it doesn't mean that the reflected wave simply turns over. During reflection the form of the wave is essentially deformed. It occurs due to the fact that each ordinate $y_n = y(t_n)$ reflects back at the different moments of time $t = t_n^*$, and only when $y_n = f(x)$. It means, that each amplitude y_n reflects not from the point $x = 0$, but from the point $x = \delta_n$, and thus the length of the string "looks" shorter, and this "truncation" depends on the amplitude of the incident wave. Now we can derive the formulae for description of both traveling waves y_+ and y_- moving in both directions.

In musical acoustics each function of time $y(t)$ may be considered as a baseband signal, with range of frequencies measured from close to 0 Hz to a highest signal frequency, which is equal to 100 kHz, approximately. Therefore, such function is completely determined by giving its ordinates as a series of discrete points [5]

$$y_-(t) = \sum_{n=-\infty}^{+\infty} y(t_n) \frac{\sin \omega_{max}(t - t_n)}{\omega_{max}(t - t_n)} . \quad (1)$$

Here $t_n = n\pi/\omega_{max}$, and ω_{max} is a maximum bandwidth of the signal.

Because each ordinate $y(t_n)$ reflects back at moment $t = t_n^*$ when $y_n = y(t_n) = f(x)$, the reflected wave can be represented in the form

$$y_+(t) = -\sum_{n=-\infty}^{+\infty} y(t_n) \frac{\sin \omega_{max}(t - t_n + t_n^*)}{\omega_{max}(t - t_n + t_n^*)} , \quad (2)$$

where $t_n^* = \frac{1}{c} f^{-1}(y_n)$. Here $f^{-1}(y_n)$ denotes the inverse function of $f(x)$.

4. PIANO STRING EXCITATION AND BASIC FORMULAE

The presented model is used here for demonstration of the influence of the contact nonlinearity and the shape of the contact surface on the spectral structure of the piano string vibration.

Let's consider an ideal (flexible) string. The displacement $y(x, t)$ of such a string obeys the simple wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \quad (3)$$

In [6], the following system of equations describing the hammer-string interaction is employed

$$\frac{dz}{dt} = -\frac{2T}{cm} g(t) + V, \quad (4)$$

$$\frac{dg}{dt} = \frac{c}{2T} F(t), \quad (5)$$

where $g(t)$ is the outgoing wave created by the hammer strike at the contact point $x = l$, $F(t)$ is the acting force; m, $z(t)$, and V are the hammer mass, the hammer displacement, and the hammer velocity, respectively. The hammer felt compression is determined by $u(t) = z(t) - y(l, t)$. Function $y(l, t)$ describes the string transverse displacement at the contact point $x = l$, and is given by [7]

$$y(l, t) = g(t) + 2\sum_{i=1}^{\infty} g\left(t - \frac{2iL}{c}\right) - \sum_{i=0}^{\infty} g\left(t - \frac{2(i+a)L}{c}\right) - \sum_{i=0}^{\infty} g\left(t - \frac{2(i+b)L}{c}\right). \quad (6)$$

Here it is assumed that the string of length L extends from $x = 0$ on the left to $x = L$. Parameter $a = l/L$ is the fractional length of the string to striking point, and $b = 1 - a$. Parameter a determines the actual distance l of the striking point from nearest string end. The initial conditions at the moment when the hammer first contacts the string, are taken as $g(0) = z(0) = 0$, and $dz(0)/dt = V$.

The hammer is described here in according to tree-parameter hereditary model that is presented in [8]. According to this model, the nonlinear force $F(t)$ exerted by the hammer is related to the felt compression $u(t)$ by the following expression

$$F(u(t)) = Q_0 \left[u^p + \alpha \frac{d(u^p)}{dt} \right] . \quad (7)$$

Here the parameter Q_0 is the static hammer stiffness; p is compliance nonlinearity exponent, and α is the retarded time parameter.

If the string has the rigid termination, the spectrum of the string motion exited by the hammer may be calculated [6] directly from the force history $F(t)$. The general expression for the string mode energy level is

$$E_i = 10 \log \left[\frac{2M\omega_i^2}{mV^2} \left(A_i^2 + B_i^2\right) \right] , \quad (8)$$

where

$$A_i = \frac{\sin(ia\pi)}{i\pi c\mu} \int_0^{t_0} F(s) \cos(\omega_i s) ds, \quad (9)$$

$$B_i = \frac{\sin(ia\pi)}{i\pi c\mu} \int_0^{t_0} F(s) \sin(\omega_i s) ds. \quad (10)$$

Here $\omega_i = \pi i c L^{-1} = i\omega_0$ is the string mode angular frequency; t_0 is the contact time. In our case the string at one end has the nonlinear support, therefore we must include this edge influence on the string vibrations by other way.

At the first stage the outgoing wave $g(t)$ generated by the hammer strike may be considered as the initial local disturbance of the string motion, which creates a sequence of pulses $g_n = g(t_n)$ satisfying the conditions of relation (1). Each pulse is reflected back according to relation (2). Thus this model of the nonlinear reflection gives possibility to obtain the distribution of the transverse displacement $y(x, t)$ of the string at any moment by the following expressions

$$y(x,t) = \sum_{i=0}^{\infty} g\left(t - T_{1i}\right) - \sum_{i=1}^{\infty} g\left(t - T_{2i}\right) \text{ , if } x \leqslant l \text{ ,} \quad (11)$$

$$y(x,t) = \sum_{i=0}^{\infty} g\left(t - T_{3i}\right) - \sum_{i=1}^{\infty} g\left(t - T_{4i}\right) \text{ , if } x > l \text{ .} \quad (12)$$

Here $g(\xi) = 0$ if $\xi \leqslant 0$, and the functions T_{ji} are given by

$$T_{1i} = c^{-1}[l - x + 2i(L - \delta_{1i})] , \quad (13)$$

$$T_{3i} = c^{-1}[x - l + 2i(L - \delta_{3i})] , \quad (14)$$

$$T_{2i} = c^{-1}[l + x - 2L + 2i(L - \delta_{2i})] , \quad (15)$$

$$T_{4i} = c^{-1}[2\delta_{4i} - x - l + 2i(L - \delta_{4i})] . \quad (16)$$

In our case $\delta_{ij} \ll L$, and we can find $\delta_{ij} = f^{-1}(G_{ij})$, where

$$G_{1i} = g(t - c^{-1}(l - x + 2iL)) , \quad (17)$$

$$G_{2i} = g(t - c^{-1}(l + x - 2L + 2iL)) , \quad (18)$$

$$G_{3i} = g(t - c^{-1}(x - l + 2iL)) , \quad (19)$$

$$G_{4i} = g(t - c^{-1}(-x - l + 2iL)) . \quad (20)$$

Using such procedure, we can define function $y(x,t_0)$ as an initial string displacement at the moment $t = t_0$, just as the string vibrates freely. The initial string velocity $v(x,t_0)$ at this moment we can find using the string displacement $y(x,t_0-\Delta)$, where $\Delta = t_i - t_{i-1} = \pi/\omega_{max}$. Then, the initial string velocity we can determine as

$$v(x,t_0) = \frac{\partial y}{\partial t}\bigg|_{t=t_0} = \frac{y(x,t_0) - y(x,t_0-\Delta)}{\Delta}. \qquad (21)$$

Now using Fourier analysis we can find the spectrum of the string vibrations. If

$$y(x,t) = \sum_i (A_i \cos\omega_i t + B_i \sin\omega_i t)\sin\left(\frac{i\pi x}{L}\right), \qquad (22)$$

with normal-mode frequencies $\omega_i = i\omega_0$, one finds

$$A_i = \frac{2}{L}\int_0^L y(x,t_0)\sin\left(\frac{i\pi x}{L}\right)dx, \qquad (23)$$

$$B_i = \frac{2}{L\omega_i}\int_0^L v(x,t_0)\sin\left(\frac{i\pi x}{L}\right)dx, \qquad (24)$$

and the string mode energy level E_i of the ith mode is also defined by Eq. (8).

5. NUMERICAL RESULTS

For numerical simulation of the piano string with nonlinear support we chose here the note number $n = 70$ (note $F_6^\sharp$, frequency $f = 1480$ Hz). The string parameters are the following: the string length $L = 119$ mm; the actual distance of the striking point from nearest string end $l = 7.2$ mm; the string tension $T = 644.8$ N.
The continuous variations in the hammer parameters vs. key number n were obtained experimentally by measuring a whole hammer set of recently produced unvoiced *Abel* hammers. The result of those experiments are presented in [8, 9, 10]. A best match to the whole set of hammers was approximated using

$$\begin{aligned} Q_0 &= 183\exp(0.045n), \\ p &= 3.7 + 0.015n, \\ \alpha &= 259.5 + 0.58n + 6.6\cdot 10^{-2}n^2 - \\ &- 1.25\cdot 10^{-3}n^3 + 1.172\cdot 10^{-5}n^4, \end{aligned} \qquad (25)$$

for hammer number $1 \leqslant n \leqslant 88$. Here the dimension of parameter α is [ms], and the dimension of Q_0 is [N/mm^p].
The hammer masses of this set were approximated by

$$m = 11.074 - 0.074n + 10^{-4}n^2, \quad 1 \leqslant n \leqslant 88. \qquad (26)$$

The mass of hammer $n = 1$ (A_0) is 11.0 g, and the mass of hammer $n = 88$ (C_8) 5.3 g.
For the hammer number $n = 70$ we use such values of parameters: static stiffness $Q_0 = 4270$ N/mm^p; nonlinearity exponent $p = 4.75$; hereditary parameter $\alpha = 0.395$ ms.
Because this note has three strings per note, the hammer mass for numerical simulation was taken as 1/3 of the real hammer mass, and is equal here $m = 2.1$ g.
The rigid edge of the frame has approximately a parabolic form, and it is described here by the function $y = (2R)^{-1}x^2$, where R is the radius of the curvature of the edge at $x = 0$.
The first step of numerical simulation of the string vibration begins with calculation of the outgoing wave $g(t)$created by the hammer strike using formulae (4 – 7). Then we can find the form of the string using relationships (11) and (12).

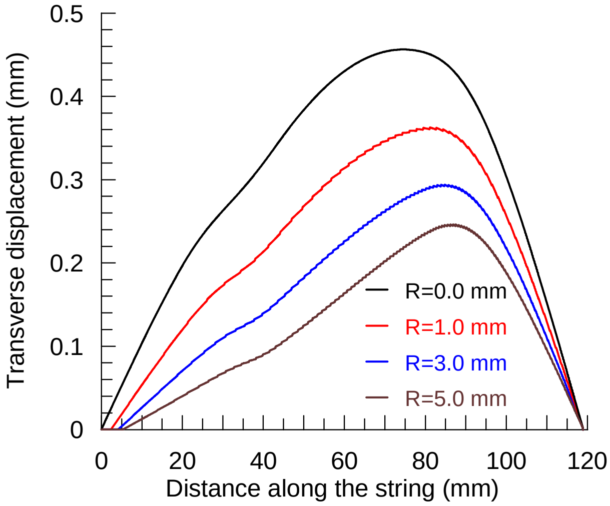

Figure 3: Transverse string displacement for string $n = 70$. Varying the edge curvature R with fixed hammer velocity $V = 3$ m/s.

In Figure 3 is presented the string transverse displacement along the string length just at the moment t_0, when the hammer has lost the contact with the string. The results are shown for different values of the curvature R of the edge of the frame, and for the fixed initial hammer velocity is $V = 3$ m/s. In this Figure the influence of the edge on the form of the string displacement is well visible. In vicinity of the edge ($x = 0$) the amplitude of the string deflection becomes smaller with increasing of the radius of the edge.
To calculate the string vibration spectrum according to relationships (8, 22 – 24), we must obtain the distribution of the string velocity according to (21). For this purpose we calculate the string displacement just one step of time Δ before the moment, when the hammer has lost the contact with the string. Here, for this note number $n = 70$ ($f = 1480$ Hz) we choose $\Delta = 0.204\ \mu$s.
Using the data about the string displacement presented above, by means of numerical differentiation (21), we can find the string velocity distribution, which is displayed in Figure 4.

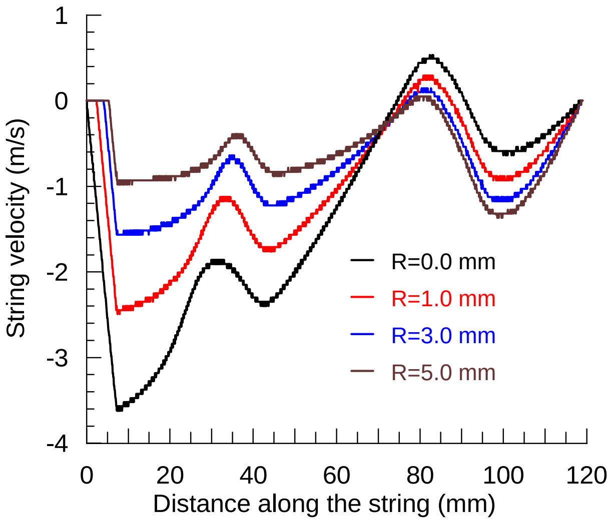

Figure 4: Velocity distribution for string $n = 70$. Varying the edge curvature R with fixed hammer velocity $V = 3$ m/s.

Here we can see that the influence of the edge curvature on the string velocity and on the string displacement is similar. In vicinity of the edge the string velocity is equal almost to zero along a distance, which becomes longer with increasing of the radius of curvature of the edge.

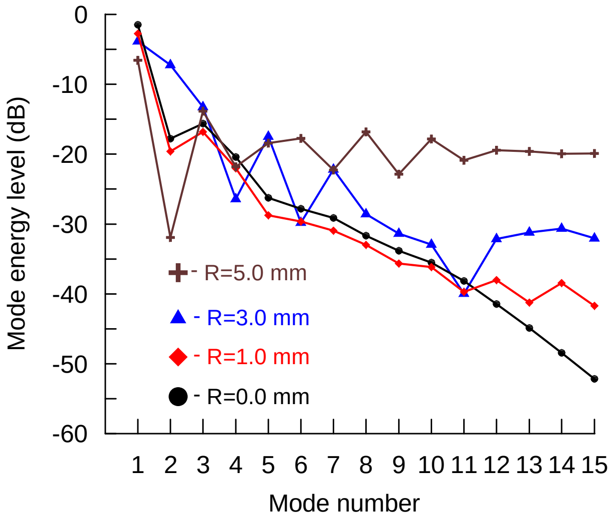

Figure 5: Spectral envelopes for string $n = 70$. Varying the edge curvature R with fixed hammer velocity $V = 3$ m/s.

In Figure 5 we demonstrate the influence of the edge curvature on the spectrum of the string vibrations excited by the hammer with initial velocity is $V = 3$ m/s.
It is clear that with increasing of the edge curvature the amplitude of higher harmonics becomes greater. Moreover, the form of the spectral envelopes for $R = 3$ mm and 5 mm is essentially irregular, and the rate of higher harmonics attenuation changes significantly. Obviously, the edge curvature $R \geqslant 5$ mm creates the train of oscillations up to very high frequencies. Finally, we can see the strong influences of the edge curvature on the amplitude of the second harmonic, and this fact is very important.

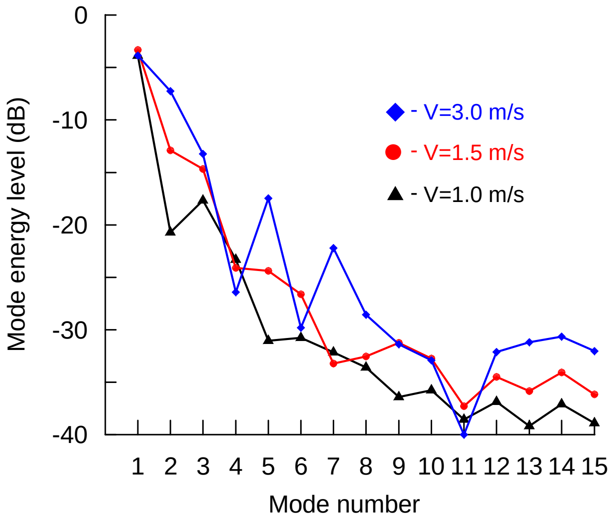

Figure 6: Spectral envelopes for string $n = 70$. Varying hammer velocity V with fixed edge curvature $R = 3$ mm.

In Figure 6 is presented the influence of the velocity of the hammer strike on the spectrum of the string vibrations.
It is evident that the power spectrum of the string vibration grows up significantly and reshapes essentially with increasing of the hammer velocity.

6. CONCLUSIONS

We have provided a careful model of piano string with nonlinear support, and found that this theory may be of some use for piano treble strings, which one end is terminated on the curved edge of the frame. One respect in which this model is still idealized is its assumption about a very simple string boundary condition at the piano bridge.
It is found that the new trains of high frequency oscillations that do not exist initially grow up eventually, and its appearance depends on the curvature of the edge of the frame. It is shown that the power spectrum of the string vibration is enriched by spectral components up to very large numbers, and essentially reshapes with increasing of the amplitude of the initial wave exited by the piano hammer.
It is revealed that even the small variation of the edge curvature significantly influenced on the amplitude of the second harmonic in fact. For this reason the manufacturers of grand pianos should produce a cast iron frame very accurately, and carefully process the surface of the edge.

7. ACKNOWLEDGEMENT

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8. REFERENCES

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