Kinematics of Ideal String Vibration Against a Rigid Obstacle

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Abstract

This paper presents a kinematic time-stepping modeling approach of the ideal string vibration against a rigid obstacle. The problem is solved in a single vibration polarisation setting, where the string's displacement is unilaterally constrained. The proposed numerically accurate approach is based on the d'Alembert formula. It is shown that in the presence of the obstacle the lossless string vibrates in two distinct vibration regimes. In the beginning of the nonlinear kinematic interaction between the vibrating string and the obstacle the string motion is *aperiodic* with constantly evolving spectrum. The duration of the aperiodic regime depends on the obstacle proximity, position, and geometry. During the aperiodic regime the fractional subharmonics related to the obstacle position may be generated. After relatively short-lasting aperiodic vibration the string vibration settles in the *periodic* regime. The main general effect of the obstacle on the string vibration manifests in the widening of the vibration spectra caused by transfer of fundamental mode energy to upper modes. The results presented in this paper can expand our understanding of timbre evolution of numerous stringed instruments, such as, the guitar, bray harp, tambura, veena, sitar, etc. The possible applications include, e.g., real-time sound synthesis of these instruments.

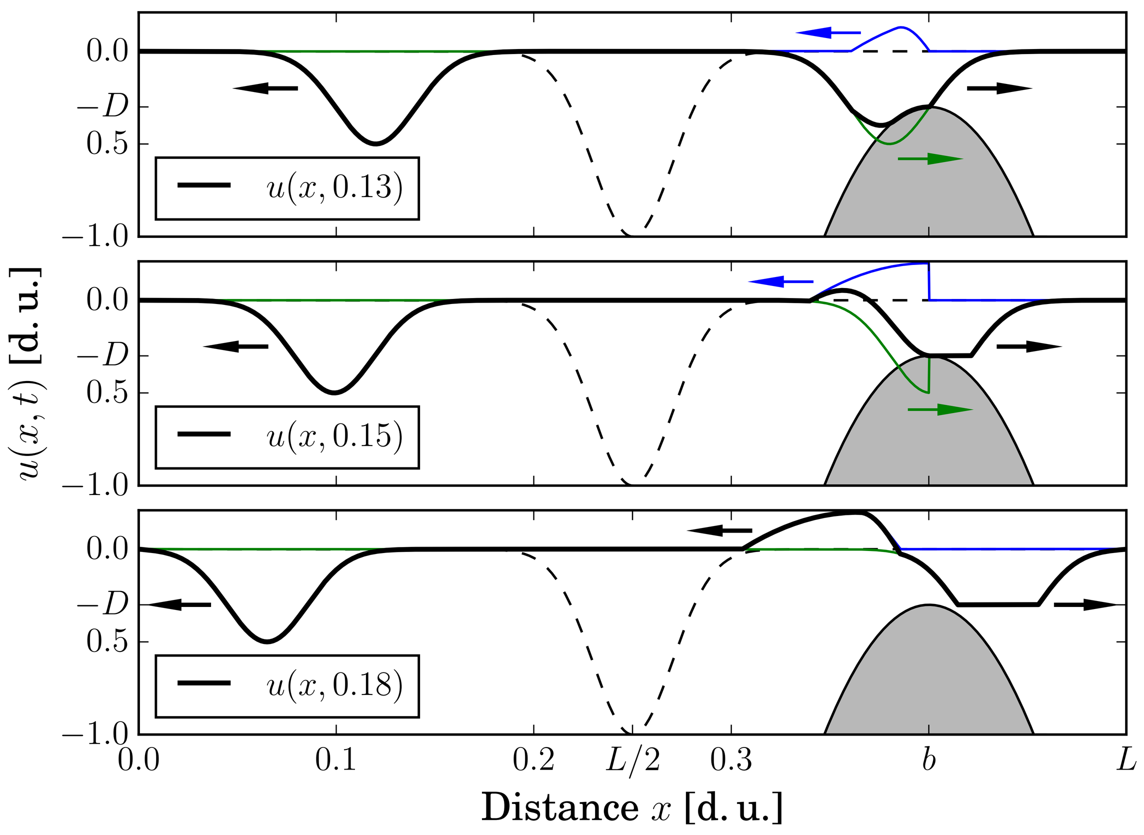

Figure 1: Reflection of travelling wave $r(x-ct)$, shown with the thin green line, from the obstacle that is shown with the grey formation. The reflected travelling wave $\hat{l}(x+ct)$ is shown with the thin blue line. Arrows indicate the directions of wave propagation. Bell-shaped initial condition is shown with the dashed line.

Model Assumptions

Vibration of lossless ideal string in a single vibration polarisation setting is described with the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u(x,t)$ is the transverse displacement of the string, $c = \sqrt{T/\mu}$ is the speed of the waves travelling on the string, where T is the tension and μ is the linear mass density of the string, having the dimension [kg/m] [1]. The vibration is numerically modeled using finite difference method which in this case is equivalent to digital waveguide modeling based on travelling wave decomposition and use of delay lines [2]. The interaction between the vibrating string and the obstacle is understood here in terms of kinematics, i.e., we study the string motion without considering its mass nor the possible inertial and reactional forces acting between the string and the obstacle during the collisions.

String–Obstacle Interaction Kinematics

Let us consider a smooth and absolutely rigid impenetrable obstacle. The obstacle is placed near the vibrating string so that it is able to obscure string's displacement $u(x,t)$. We select an obstacle with parabolic cross-section profile

$$B(x) = -[(x-b)^2/(2R) + D], \tag{2}$$

where $x = b$ is the position of the obstacle along the string, D is the vertical proximity of the obstacle to the string at its rest position, and R is the curvature radius of parabola at its apex.

Reflection of Travelling Wave $r(x-ct)$

The kinematic modeling of the string–obstacle interaction is a twofold problem. First, one needs to consider travelling wave $r(x-ct)$ approaching the obstacle form the left side. Second, one considers the travelling wave $l(x+ct)$ approaching the obstacle from the right side.

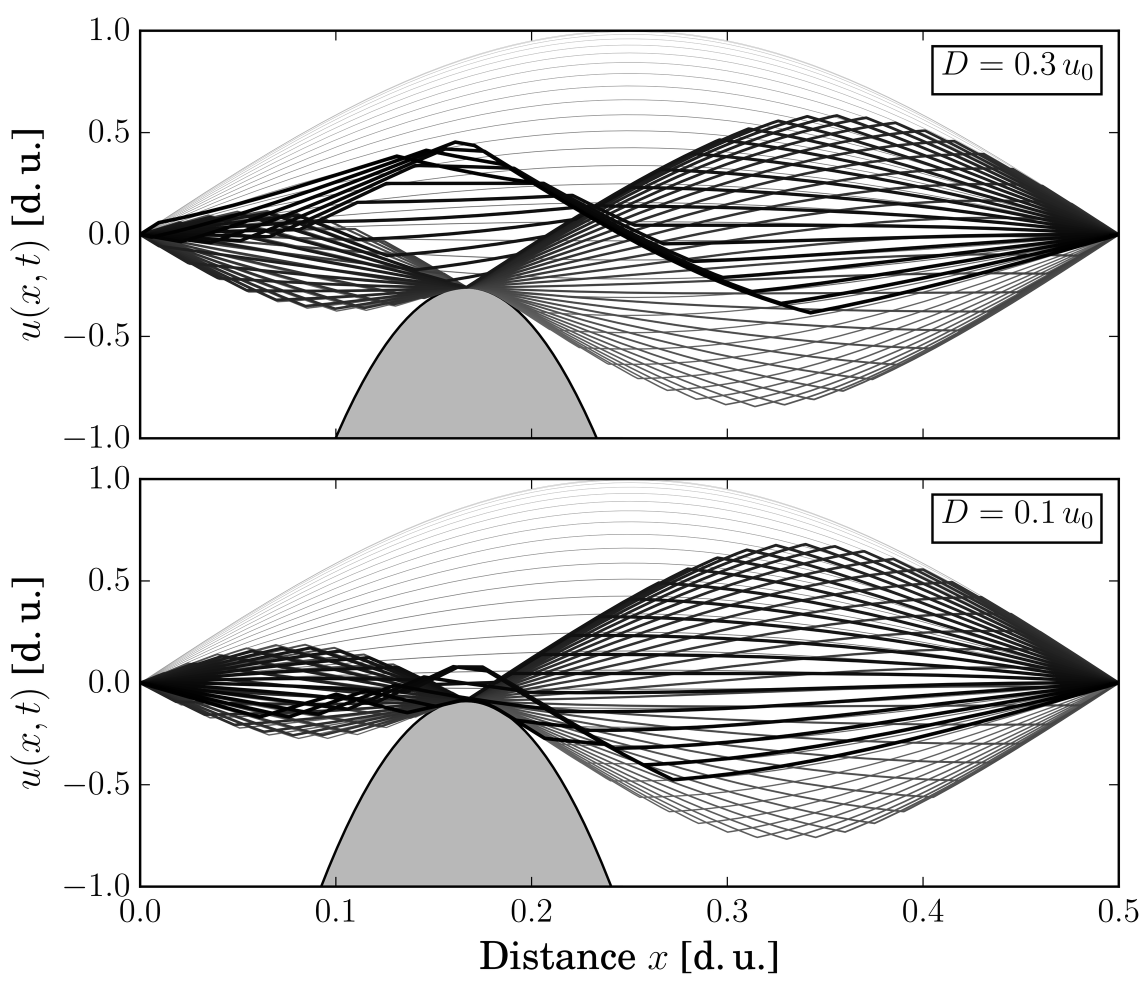

Figure 2: Stroboscopic plot of the string displacement during the first period of vibration where $t \in [0, P]$. Showing 68 time steps. The thickness of the lines is proportional to the direction of time flow. The obstacle with radius $R = 3 \cdot 10^{-3}$ is positioned at $b = L/3$. The obstacle with proximity $D = 0.3\,u_0(b)$ (top) and $D = 0.1\,u_0(b)$ (bottom) are presented.

The heuristics of the following approach is strictly determined by the d'Alembert formula. Any change in string displacement $u(x,t)$ imposed by the obstacle must involve both travelling waves. The reflection of the travelling wave $r(x-ct)$ approaching the obstacle from the left is determined by a change in the travelling wave $l(x+ct)$. We assume that the travelling wave $l(x+ct)$ resulting from the interaction first appears, or already existing one is modified, only at the point $x = x^* \leq b$, where the amplitude of string displacement $u(x^*,t) < B(x^*)$, i.e., the string attempts to penetrate the obstacle. The position of this point x^* is determined by the obstacle profile geometry $B(x)$. Because, the obstacle is in fact impenetrable we must have $u(x^*,t) = B(x^*)$ at the moment of collision. This condition determines the shape of a *reflected* travelling wave

$$\hat{l}(x^*,t) = B(x^*) - u(x^*,t). \tag{3}$$

Once one has determined the shape of the *reflected* travelling wave (3) for given time moment. It is used inside the same time moment as a travelling wave $l(x+ct)$ or to modify the already existing travelling wave moving to the left in the following manner:

$$l(x^*,t) = l(x^*,t) + \hat{l}(x^*,t). \tag{4}$$

The above modification of the travelling wave $l(x+ct)$ simply ensures that the resulting string displacement $u(x^*,t) = r(x^*,t) + l(x^*,t)$ does not geometrically penetrate the obstacle during the collision. Figure 1 shows the evolution of reflection of a pulse and its travelling waves from the obstacle. Determination of reflection of travelling wave $l(x+ct)$ approaching the obstacle from the right side is the corresponding symmetric problem with symmetry axis at $x = b$.

Results

Figure 2 shows the influence of varying the value of the obstacle proximity D on the string shape $u(x,t)$ for the duration of the first period only. The results are shown for initial condition $u_0(x) = u(x,0) = A\sin 2\pi x$, where $A = 1$. The results of frequency domain analysis of the vibration are shown in Fig. 3.

Visual inspection of the time series $u(x_r,t)$ reveals that the string vibrates in two distinct vibration regimes. The initial strong influence of the obstacle is manifested in the constantly evolving spectrum that prolongs for $t = t_p$. We call this regime the *aperiodic* regime. After $t = t_p$ the string vibration settles in the *periodic* regime, where the spectrum remains constant. Time moment t_p corresponds to the latest time instance where point x^* is determined. Inspection of the aperiodic regime of both cases shows that a short-lasting large amplitude fractional subharmonic at $f = 1.5$ is generated. This frequency is related to the fact that the obstacle is promoting a node at $b = L/3$ by effectively dividing the string into two vibrating segments. This partial does not survive the aperiodic regime.

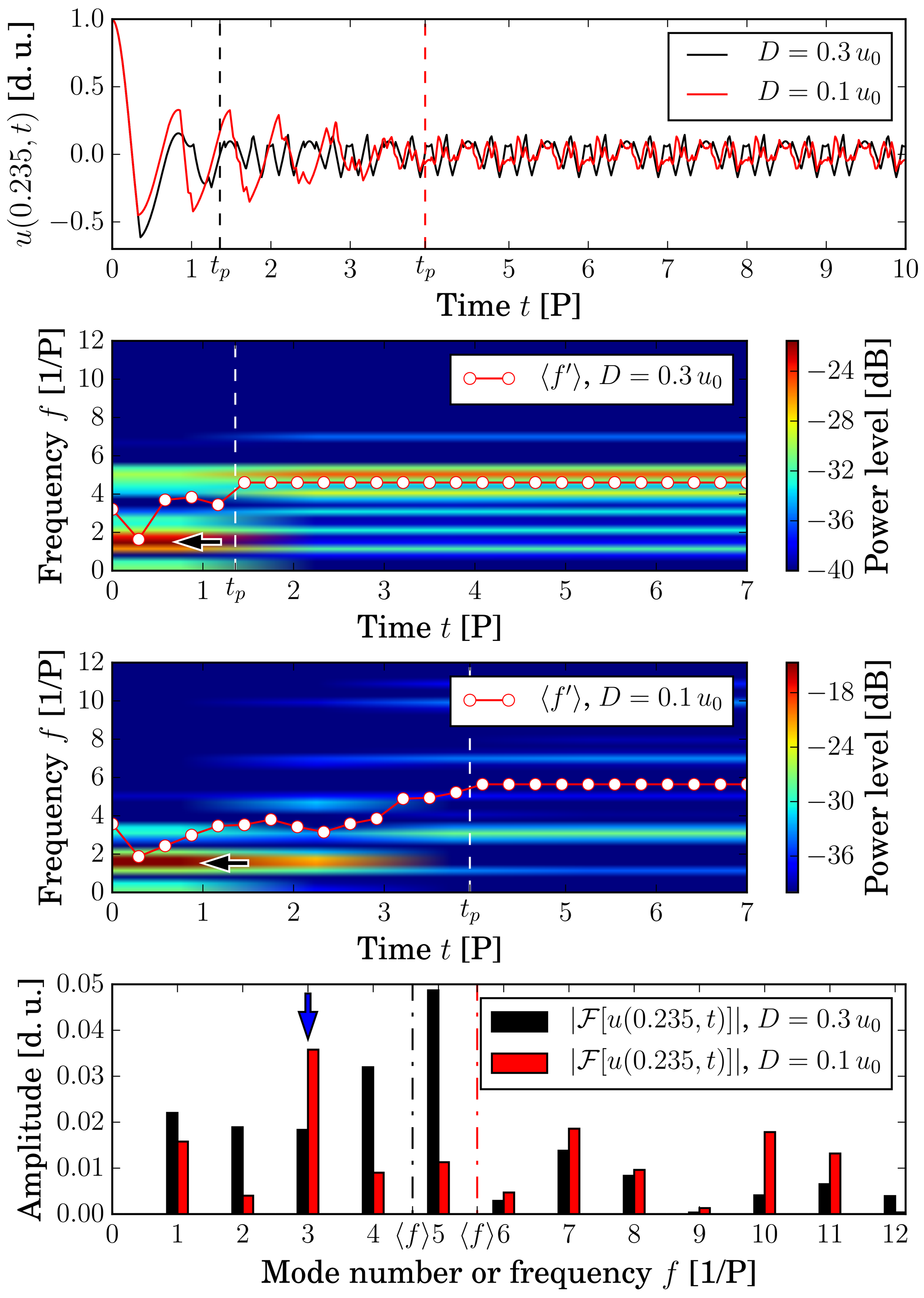

Figure 3: Top: Time series $u(x_r,t)$ shown for two values of the obstacle proximity D. Onset time t_p of the periodic regime shown with the colour-coded dashed line. Middle: Power spectrograms. Instantaneous spectral centroid $\langle f' \rangle$ is shown with the solid red line marked with bullets. Onset time t_p of the periodic regime is shown with the dashed white line. Subharmonic at $f = 1.5$ is shown with the bold arrow. Bottom: Amplitude spectra in the periodic regime where $t \in [\lceil t_p \rceil, t_{\max}]$. Spectral centroid $\langle f \rangle$ is shown with the colour-coded dash-dotted line.

Conclusions

It was shown that the ideal lossless string interacting with the obstacle vibrates in two distinct vibration regimes. In the beginning of the kinematic interaction between the vibrating string and the obstacle the string motion is *aperiodic* with constantly evolving spectrum. After some time of aperiodic vibration the string vibration settles in the *periodic* regime where the string motion is repetitious in time. The duration of the aperiodic regime depends on the obstacle proximity D, position b, and geometry (curvature radius R). The comparison of the resulting spectra in the periodic regime with the linear case where the obstacle was missing showed that the general effect of the obstacle manifests in the widening of the spectrum caused by transfer of fundamental mode energy to upper modes. The analysis of the relatively short-lasting aperiodic regime showed that the obstacle position b may generate temporary fractional subharmonics related to the node point at $x = b$. In conclusion, the results presented in this paper can expand our understanding of timbre evolution of numerous stringed instruments, such as, the guitar, bray harp, tambura, veena, sitar, etc.

References

[1] P. M. Morse, K. U. Ingard, *Theoretical Acoustics*, Princeton University Press, Princeton, New Jersey, 2016

[2] J. O. Smith, *Physical Audio Signal Processing for Virtual Musical Instruments and Audio Effects*, W3K Publishing, 2010

URL of the accompanying web-page of this paper:

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