Lecture №8: Quasi-periodicity, 3-D and higher order systems, introduction to chaos, chaotic water wheel, the Lorenz attractor, *coursework requirements*

Dmitri Kartofelev, PhD

Tallinn University of Technology,
School of Science, Department of Cybernetics, Laboratory of Solid Mechanics

Lecture outline

- Quasi-periodicity in 2-D and 3-D systems, trajectories on the surface of a torus
- 3-D systems and higher order systems
- What is chaos? Deterministic chaos, chaos theory, definition of chaos
- Examples of *chaotic* systems:
    - Chaotic water wheel, the Lorenz mill
    - The Lorenz attractor
- A remark on plotting 3-D phase portraits
- *Coursework requirements*

Quasi-periodicity

Let's consider a system[1] given in the following form:

$$\begin{cases} \dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1), \\ \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2), \end{cases} \tag{1}$$

where θ_1 and θ_2 are the angular displacements, ω_1 and ω_2 are the natural frequencies, and K_1 and K_2 are the coupling constants. Solution trajectories of Sys. (1) are studied on the surface of a torus.

[1] See Mathematica .nb file uploaded to the course website.

Quasi-periodicity[2]

Transitioning from 2-D to 3-D systems.

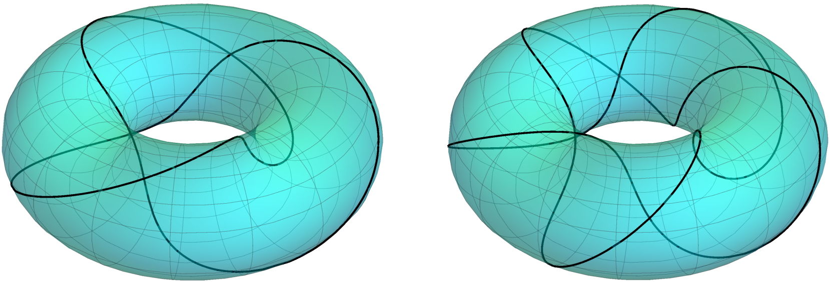


Figure: Two examples of closed trajectories defined by Sys. (1) as they appear on the surfaces of tori. (Left) Trefoil knot ($p = 3$, $q = 2$). (Right) Cinquefoil knot ($p = 5$, $q = 2$).

[2]See Mathematica .nb file uploaded to the course website.

Chaos, dictionary definition

Chaos[3] in day-to-day laymen jargon (colloquial meaning):

- a state of utter confusion or disorder.
- a total lack of organisation or order.
- complete confusion and disorder; *a state in which behaviour and events are not controlled by anything.*
- a state of things in which chance is supreme; *especially, the confused unorganised state of primordial matter before the creation of distinct forms.*
- any confused, disorderly mass: *a chaos of meaningless phrases.*

[3] Source: various online dictionaries.

Chaos in mathematics and physics

Chaos theory is the field of study in mathematics that studies the behaviour of dynamical systems that are *highly* sensitive to initial conditions – a response popularly referred to as the "**butterfly effect**". Small differences in initial conditions (such as those due to rounding errors in numerical computation or measurement uncertainty) yield widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general. This happens even though these systems are *deterministic*, meaning that their future behaviour is fully determined by their initial conditions, with no random (stochastic) elements involved. In other words, the *deterministic nature* of these systems does not make them *predictable*. This behaviour is known as **deterministic chaos**, or simply **chaos**. Chaotic behaviour exists in many natural systems, such as weather and climate. It also occurs spontaneously in some systems with artificial components, such as road traffic.

Chaos in mathematics and physics

The fact that deterministic system is not predictable (determined) in *practice* is not an internally contradicting statement, its a manifestation of a **new mathematical property or type of solution** of higher order (order more than two) nonlinear systems, called **chaos**. Also, this long-term **aperiodic** solution is qualitatively different from the periodic and quasi-periodic solutions since solutions with slightly different initial conditions deviate exponentially.

The **chaos** was summarised by Edward Lorenz as:
Chaos – when the present determines the future, but the approximate present does not approximately determine the future.

When predicting *distant* future states of a chaotic system one can never know the starting point accurately enough.

Chaos in mathematics and physics

The **chaos theory** explains deterministic systems which in principle can be predicted, for a time, then appear to become random. The amount of time patterns can be predicted depends on a time scale (the Lyapunov time) determined by the system's dynamics.

Chaos is a property of reality that we sense when we try to predict *distant* future.

SRB measure (Sinai-Ruelle-Bowen measure) – If statistics of trajectories of a system are **insensitive** to initial conditions or small differences of initial conditions then we say that the system has a SRB measure.

Chaotic systems: The Lorenz mill[4]

Equations of motion of a chaotic water wheel a.k.a. the **Lorenz mill** are the following:

$$\begin{cases} \dot{a}_1 = \omega b_1 - K a_1, \\ \dot{b}_1 = -\omega a_1 - K b_1 + q_1, \\ \dot{\omega} = -\dfrac{\nu}{I}\omega + \dfrac{\pi G r}{I} a_1, \end{cases} \tag{2}$$

where I is the moment of inertia, θ is the angle of the wheel, ω is the angular velocity, K is the liquid's leakage rate, ν is the damping rate, r is the radius of the wheel, G is the effective gravity constant. a_1 and b_1 are the Fourier amplitudes of the first modes of the liquid's mass distribution function

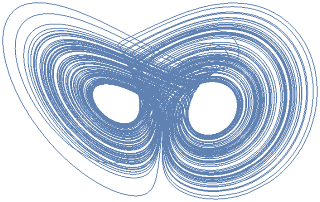
$$m(\theta, t) = \sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta. \tag{3}$$

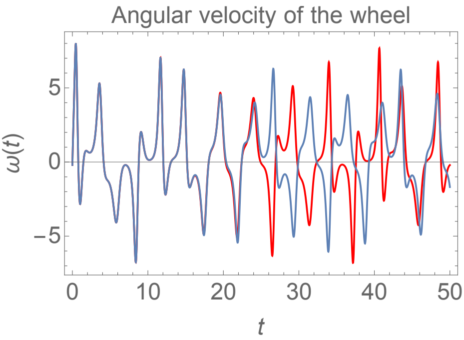
[4]See Mathematica .nb file uploaded to the course website.

Chaotic systems: The Lorenz mill

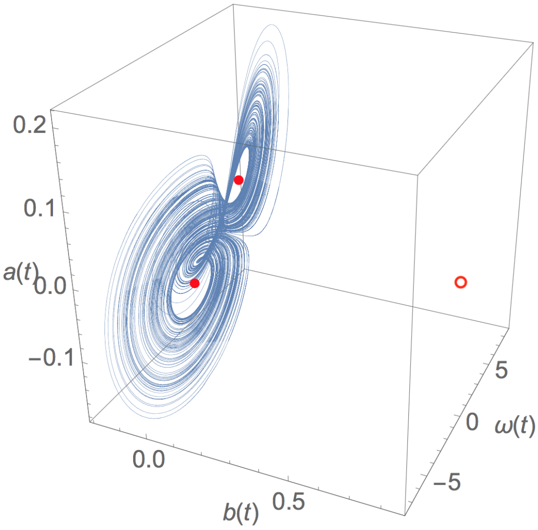
And, g_1 is the Fourier amplitude of the first mode of the liquid inflow mass distribution function

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta. \tag{4}$$



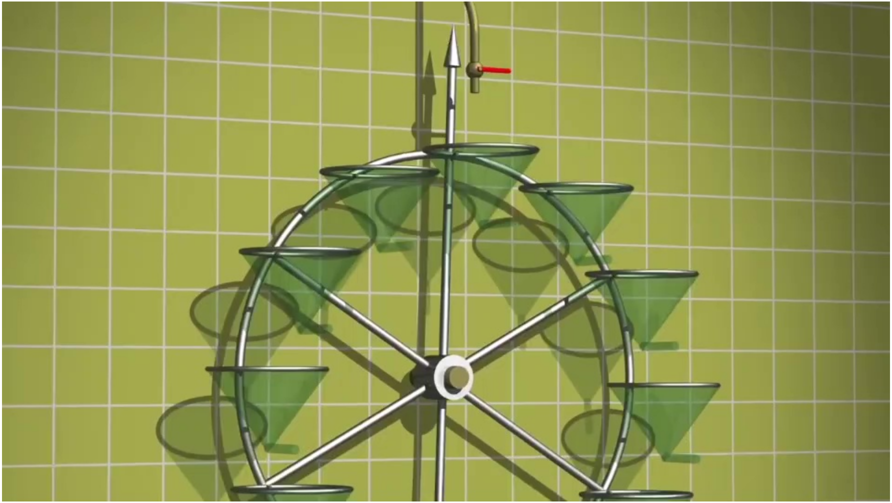


Chaotic systems: The Lorenz mill[5]



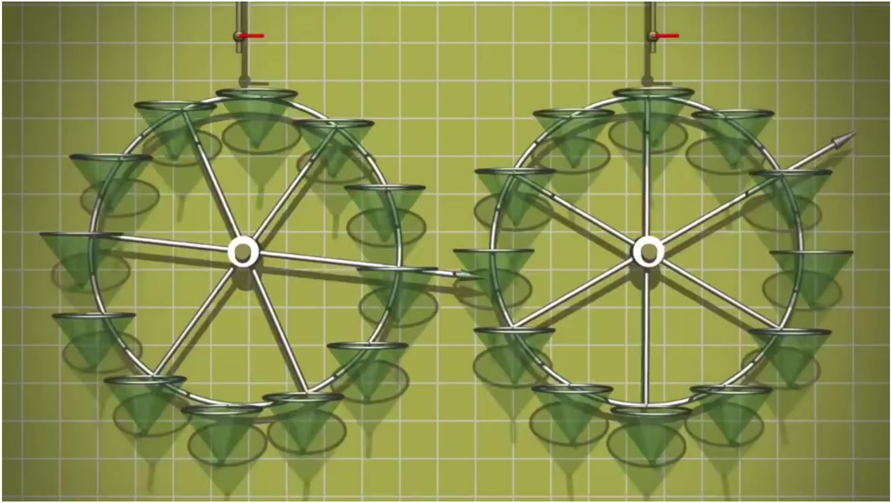
[5]See Mathematica .nb file uploaded to the course website.

Chaotic systems: The Lorenz mill and chaos



Credit:

Chaotic systems: The Lorenz mill, SRB measure



Credit: CC-BY: Jos Leys, Étienne Ghys, Aurélien Alvarez, `http://www.chaos-math.org/`

Chaotic systems: The Lorenz attractor

The **Lorenz attractor:**[6] It can be shown that Sys. (2) is a specific case of a more general system in the form:

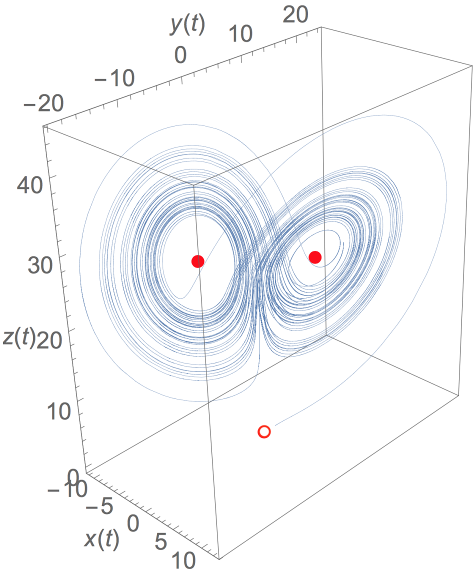
$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{5}$$

where σ, r, $b > 0$ are the control parameters.

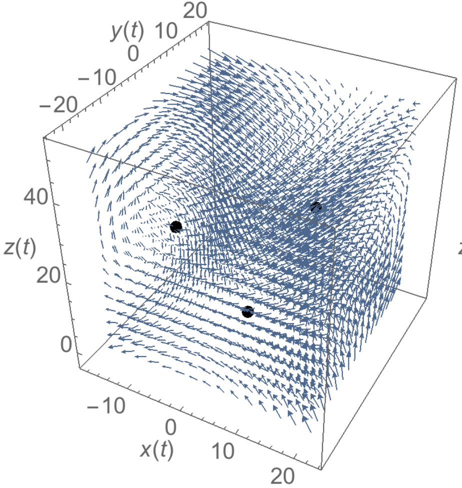
Read: E. N. Lorenz, "Deterministic nonperiodic flow". *Journal of the Atmospheric Sciences*, **20**(2), pp. 130–141 (1963). `http://dx.doi.org/10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2`

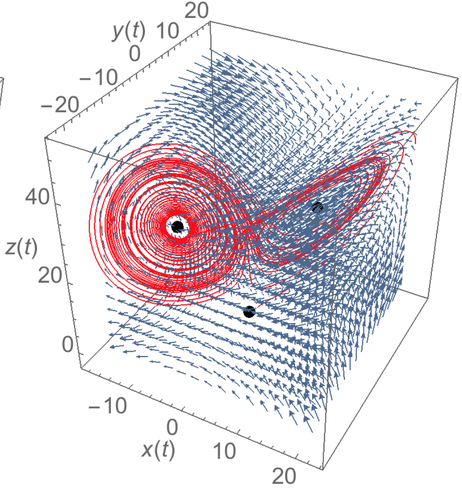
[6]See Mathematica .nb file uploaded to the course website.

Chaotic systems: The Lorenz attractor



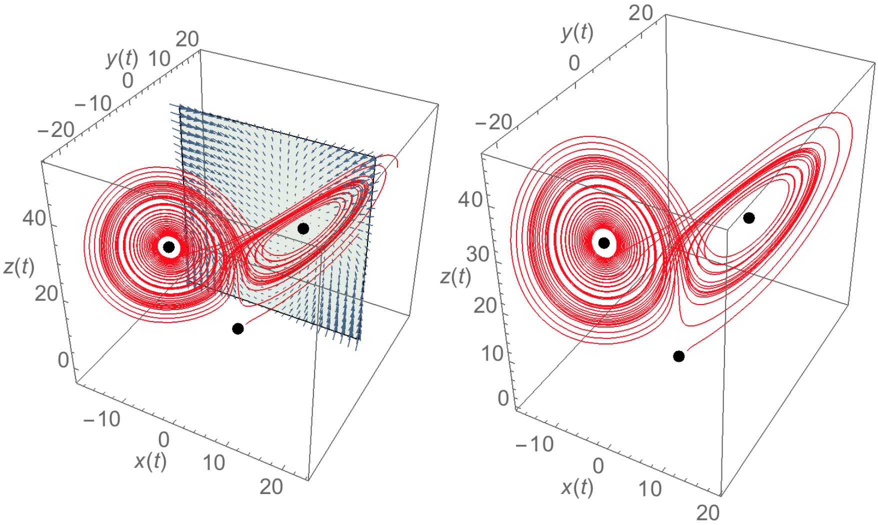
A remark on 3-D phase portrait visualisation[7]





[7] See Mathematica .nb file uploaded to the course website.

A remark on 3-D phase portrait visualisation



Conclusions

- Quasi-periodicity
- Chaos, deterministic chaos, chaos theory
- Examples of chaotic systems:
  - Chaotic water wheel, the Lorenz mill
  - The Lorenz attractor
- A remark on plotting 3-D phase portraits
- *Coursework requirements*

Revision questions

- What is quasi-periodicity?
- Can quasi-periodic system generate a chaotic solution? Why?
- Do limit-cycles exist in 3-D phase spaces? Sketch an example.
- What are 3-D and higher order systems?
- What is chaos in the context of dynamical systems (deterministic chaos, chaos theory)?
- Name properties of chaotic systems.
- What does it mean that a chaotic system has a SRB measure (Sinai-Ruelle-Bowen measure)?
- What is chaotic water wheel?
- What is the Lorenz attractor?