Lecture №7: Bifurcations in 2-D, bifurcations of fixed points, the Hopf bifurcation, bifurcations of closed orbits, examples of dynamical instabilities

Dmitri Kartofelev, PhD

Tallinn University of Technology,
School of Science, Department of Cybernetics, Laboratory of Solid Mechanics

TAL TECH
SCHOOL OF SCIENCE

Lecture outline

- Classification of bifurcations in 2-D systems
- Bifurcations of fixed points
- Bifurcations of closed orbits
- The Hopf bifurcation
- Hysteresis on level of cycles
- Videos of examples of dynamical instabilities (engineering, chemistry, neurology)

Classification of bifurcations in 2-D

Case I Bifurcations of fixed points

A) Bifurcations at $\lambda = 0$

1) Saddle-node bifurcation
2) Transcritical bifurcation
3) Pitchfork bifurcation
   - Supercritical pitchfork bifurcation
   - Subcritical pitchfork bifurcation

B) Hopf bifurcations, bifurcations at $\lambda = \pm i\omega$

1) Supercritical Hopf bifurcation
2) Subcritical Hopf bifurcation

Case II Bifurcations of closed orbits

A) Saddle-node coalescence of cycles (accompanied by subcritical Hopf)

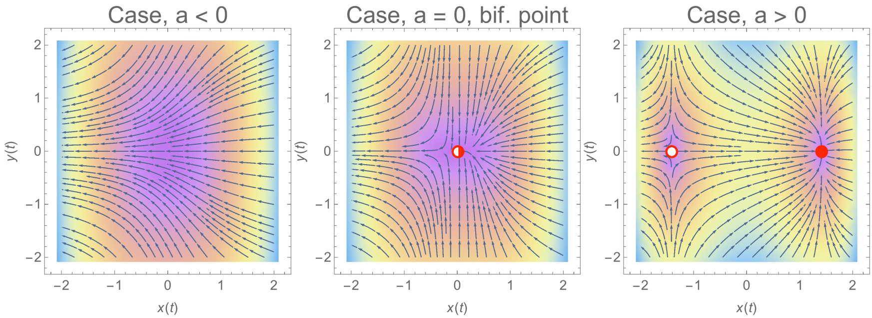
B) SNIPER (saddle-node infinite period bif.) or SNIC (saddle-node in invariant cycle bif.)

C) Homoclinic bifurcation or saddle-loop bifurcation

Case I: Bifurcations of fixed points

Case I A 1: Saddle-node bifurcation. Normal form:

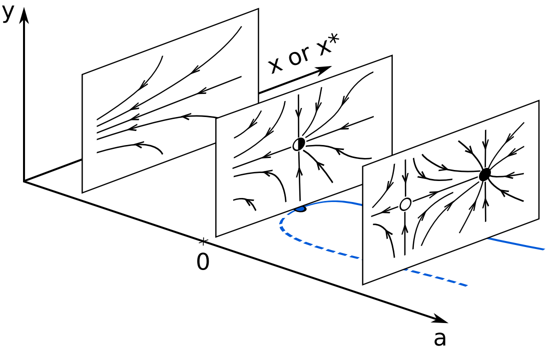
$$\begin{aligned} \dot{x} &= a - x^2 \\ \dot{y} &= -y \end{aligned} \tag{1}$$

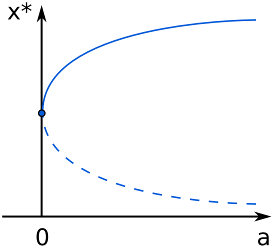


Case I: Bifurcations of fixed points

Case I A 1: Saddle-node bifurcation. Normal form:

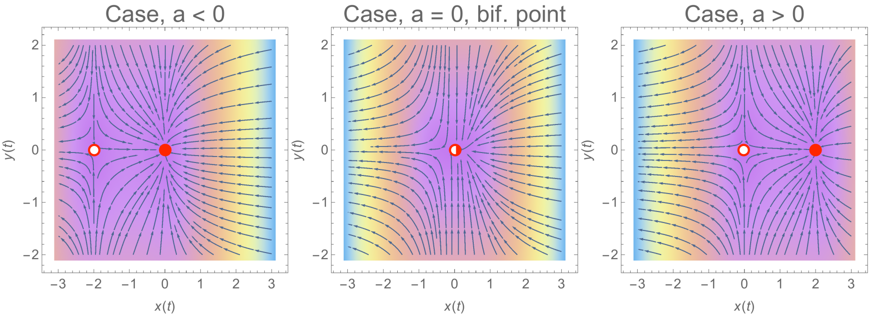
$$\dot{x} = a - x^2$$
$$\dot{y} = -y$$





Case I A 2: Transcritical bifurcation. Normal form:

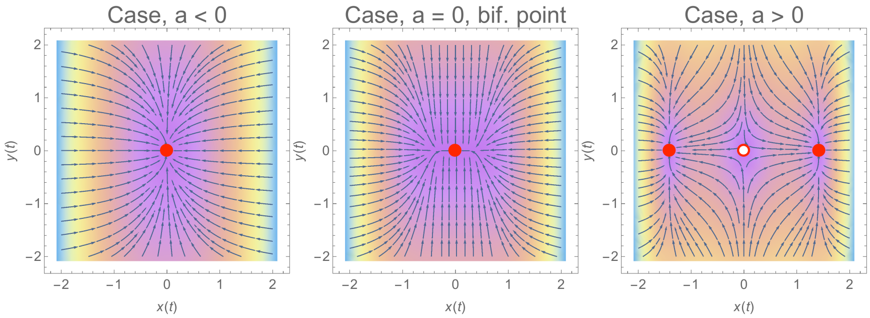
$$\dot{x} = ax - x^2$$
$$\dot{y} = -y \qquad (2)$$



Case I: Bifurcations of fixed points

Case I A 3: Supercritical pitchfork bifurcation. Normal form:

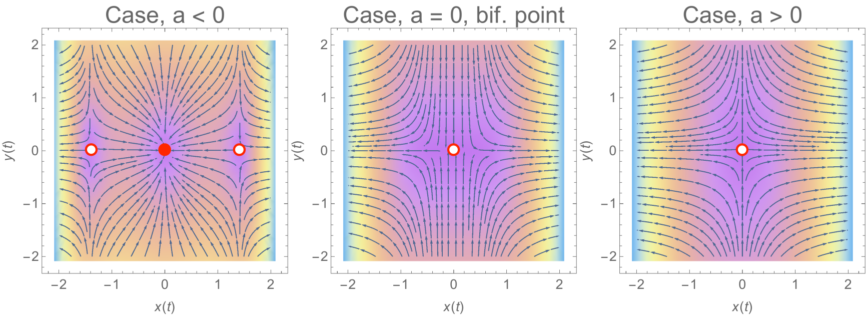
$$\begin{aligned} \dot{x} &= ax - x^3 \\ \dot{y} &= -y \end{aligned} \tag{3}$$



Case I: Bifurcations of fixed points

Case I A 3: Subcritical pitchfork bifurcation. Normal form:

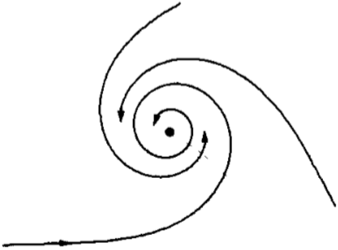
$$\begin{aligned} \dot{x} &= ax + x^3 \\ \dot{y} &= -y \end{aligned} \tag{4}$$



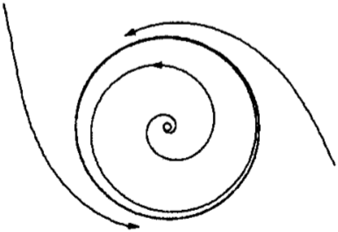
Case I: Bifurcations of fixed points

Case I B 1: The supercritical Hopf bifurcation. Bif. parameter is μ.
Normal form:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + br^2 \end{cases} \tag{5}$$



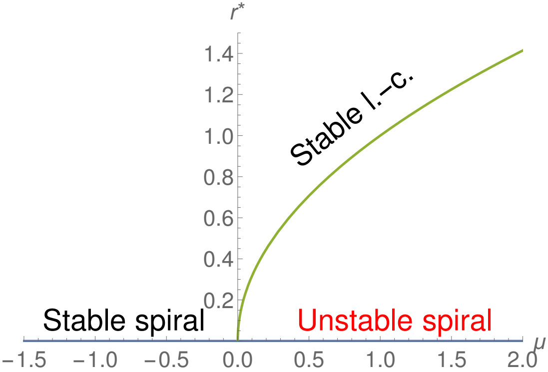
$\mu < 0$



$\mu > 0$

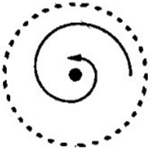
Case I B 1: The supercritical Hopf bifurcation. Bif. parameter is μ.
Normal form:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$

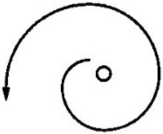


Case I B 2: The subcritical Hopf bifurcation. Bif. parameter is μ.
Normal form:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + b r^2 \end{cases} \tag{6}$$



$\mu < 0$

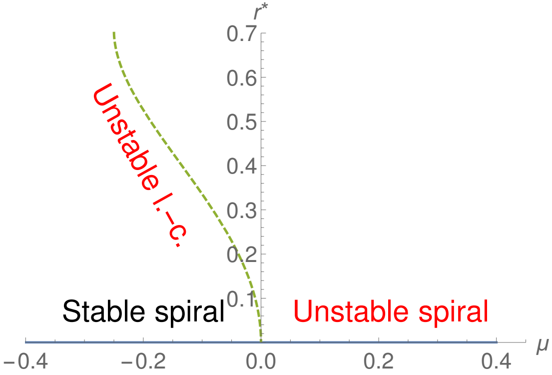


$\mu > 0$

Case I: Bifurcations of fixed points

Case I B 2: The subcritical Hopf bifurcation. Bif. parameter is μ.
Normal form:

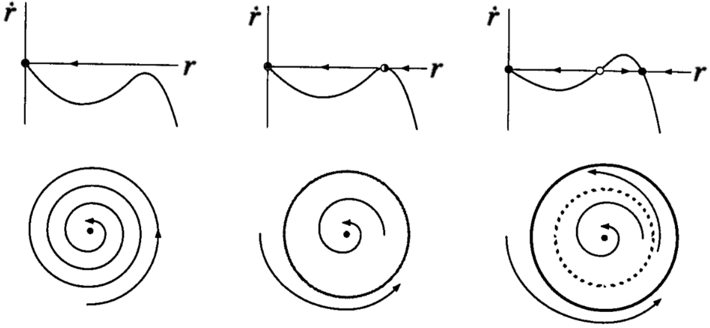
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \tag{7}$$

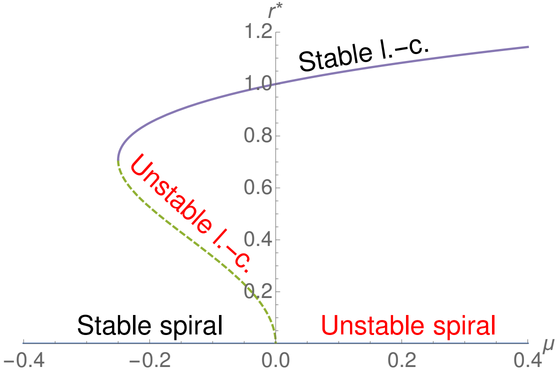


$\mu < \mu_c$ $\mu = \mu_c$ $0 > \mu > \mu_c$

Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

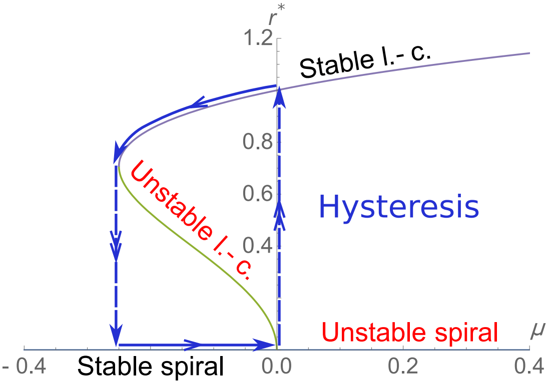
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Aeroelastic flutter

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The Tacoma Narrows bridge collapse, 1940

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The Briggs–Rauscher oscillating reaction

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The Belousov–Zhabotinsky reaction

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The tremor-dominant Parkinson's disease

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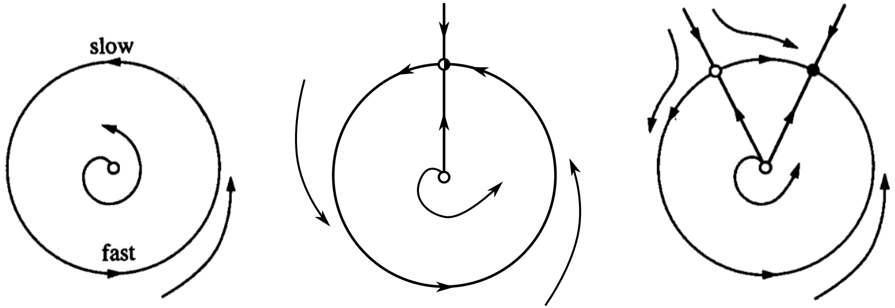
Case II: Bifurcations of closed orbits

Case II B: SNIPER or SNIC. Bif. parameter is ω. Example system:

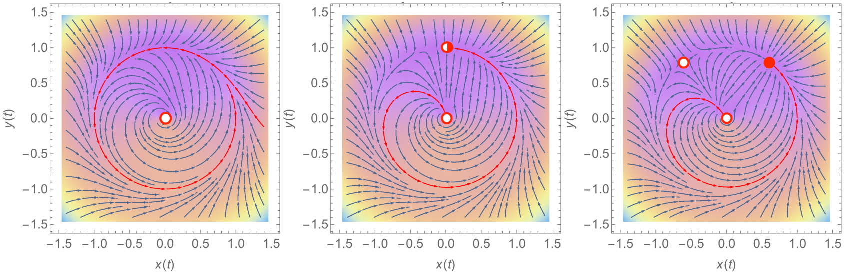
$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= \omega - \sin\theta\end{aligned} \qquad (8)$$



$\omega > 1$ $\qquad$ $\omega = 1$ $\qquad$ $\omega < 1$

Case II B: SNIPER or SNIC. Bif. parameter is ω. Example system:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \omega - \sin\theta$$

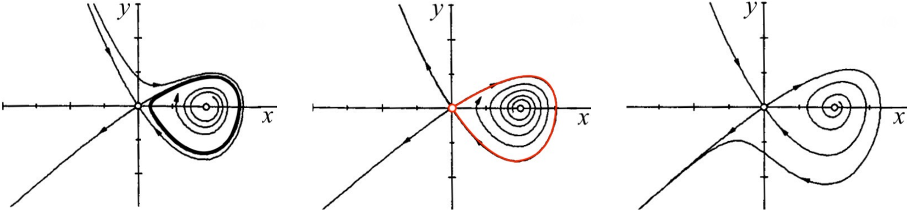


$\omega > 1$ $\qquad$ $\omega = 1$ $\qquad$ $\omega < 1$

Case II: Bifurcations of closed orbits

Case II C: Homoclinic bifurcation. Bif. parameter is μ and $\mu_c \approx -0.8645$. Example system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu y + x - x^2 + xy \end{cases} \tag{9}$$



$\mu < \mu_c$ $\mu = \mu_c$ $\mu > \mu_c$

Conclusions

- Classification of bifurcations in 2-D systems
- Bifurcations of fixed points
- Bifurcations of closed orbits
- The supercritical vs. subcritical Hopf bifurcations
- Dangers associated with the *Hopf* bifurcation
- Hysteresis on level of cycles

Revision questions

- Classification of bifurcations in 2-D.
- What is the Hopf bifurcation?
- What is the supercritical Hopf bifurcation?
- What is the subcritical Hopf bifurcation?
- What are global bifurcations of closed orbits?
- Name some global bifurcations of closed limit-cycles.
- What is a saddle-node coalescence (or bifurcation) of limit-cycles?
- What is hysteresis on level of cycles?
- Name dangers associated with the *Hopf* bifurcation.
- What is a saddle-node infinite period bifurcation?
- What is a (saddle-loop or) homoclinic bifurcation?
- Name examples of dynamical instabilities.