

Lecture №7: Bifurcations in 2-D, bifurcations of fixed points, the Hopf bifurcation, bifurcations of closed orbits, examples of dynamical instabilities

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Lecture outline

- Classification of bifurcations in 2-D systems
- Bifurcations of fixed points
- Bifurcations of closed orbits
- The Hopf bifurcation
- Hysteresis on level of cycles
- Videos of examples of dynamical instabilities (engineering, chemistry, neurology)

Classification of bifurcations in 2-D

Case I Bifurcations of fixed points

A) Bifurcations at $\lambda = 0$

- 1) Saddle-node bifurcation
- 2) Transcritical bifurcation
- 3) Pitchfork bifurcation
 - Supercritical pitchfork bifurcation
 - Subcritical pitchfork bifurcation

B) Hopf bifurcations, bifurcations at $\lambda = \pm i\omega$

- 1) Supercritical Hopf bifurcation
- 2) Subcritical Hopf bifurcation

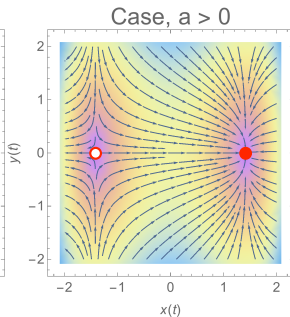
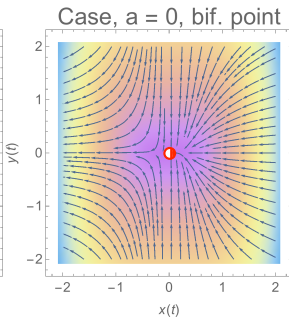
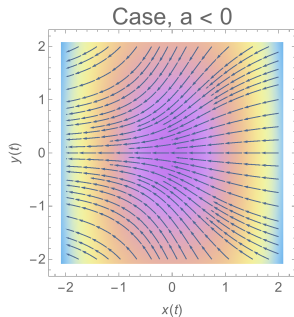
Case II Bifurcations of closed orbits

- A) Saddle-node coalescence of cycles (accompanied by subcritical Hopf)
- B) SNIPER (saddle-node infinite period bif.) or SNIC (saddle-node in invariant cycle bif.)
- C) Homoclinic bifurcation or saddle-loop bifurcation

Case I: Bifurcations of fixed points

Case I A 1: Saddle-node bifurcation. Normal form:

$$\begin{aligned}\dot{x} &= a - x^2 \\ \dot{y} &= -y\end{aligned}\tag{1}$$

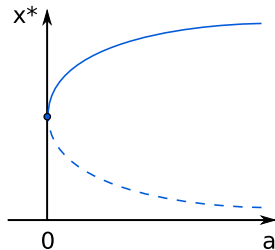
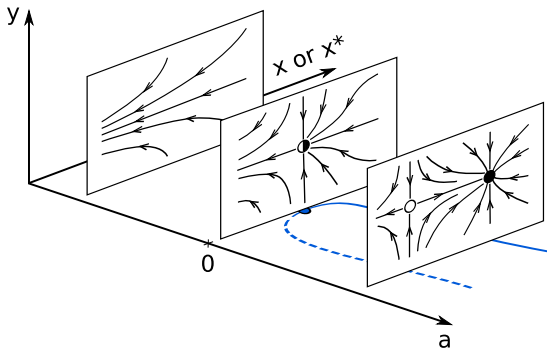


Case I: Bifurcations of fixed points

Case I A 1: Saddle-node bifurcation. Normal form:

$$\dot{x} = a - x^2$$

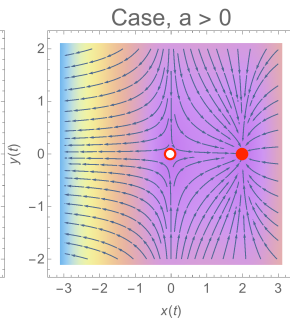
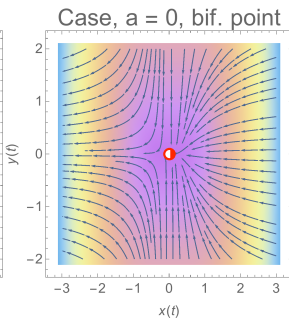
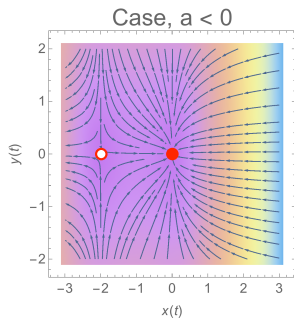
$$\dot{y} = -y$$



Case I: Bifurcations of fixed points

Case I A 2: Transcritical bifurcation. Normal form:

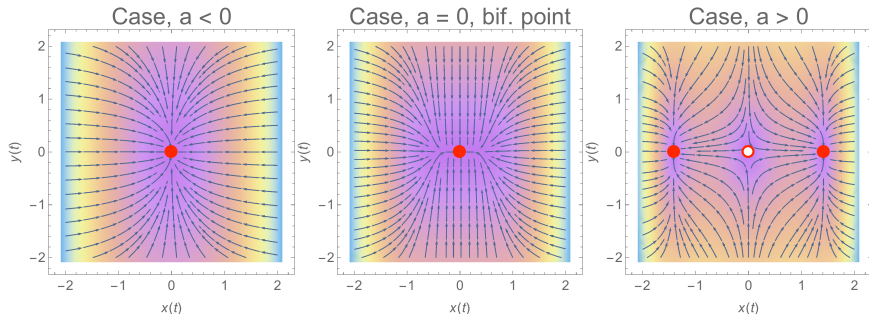
$$\begin{aligned}\dot{x} &= ax - x^2 \\ \dot{y} &= -y\end{aligned}\tag{2}$$



Case I: Bifurcations of fixed points

Case I A 3: Supercritical pitchfork bifurcation. Normal form:

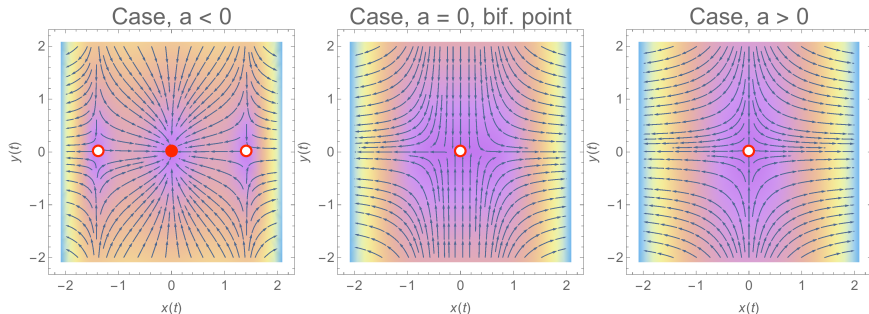
$$\begin{aligned}\dot{x} &= ax - x^3 \\ \dot{y} &= -y\end{aligned}\tag{3}$$



Case I: Bifurcations of fixed points

Case I A 3: Subcritical pitchfork bifurcation. Normal form:

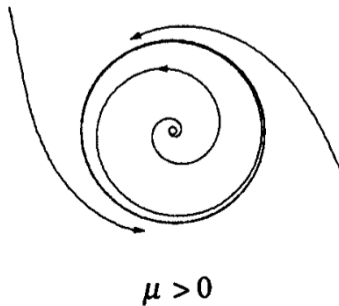
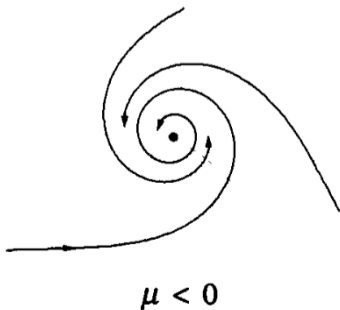
$$\begin{aligned}\dot{x} &= ax + x^3 \\ \dot{y} &= -y\end{aligned}\tag{4}$$



Case I: Bifurcations of fixed points

Case I B 1: Supercritical Hopf bifurcation. Bif. parameter is μ .
Normal form:

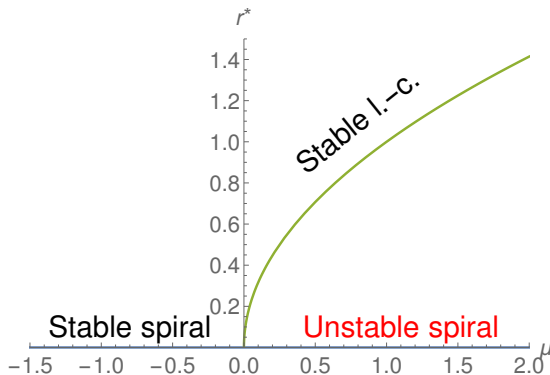
$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + br^2 \end{cases} \quad (5)$$



Case I: Bifurcations of fixed points

Case I B 1: The supercritical Hopf bifurcation. Bif. parameter is μ .
Normal form:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + br^2 \end{cases}$$

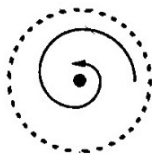


Case I: Bifurcations of fixed points

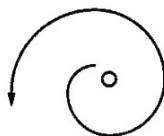
Case I B 2: The subcritical Hopf bifurcation. Bif. parameter is μ .

Normal form:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \quad (6)$$



$\mu < 0$



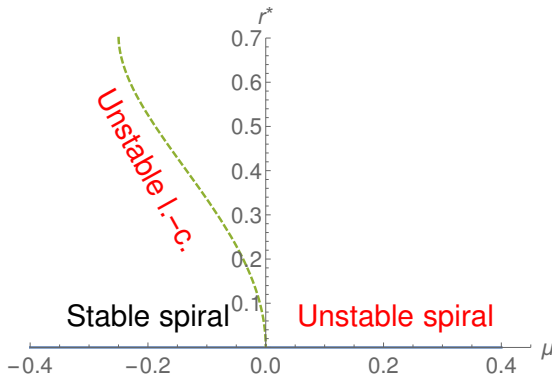
$\mu > 0$

Case I: Bifurcations of fixed points

Case I B 2: The subcritical Hopf bifurcation. Bif. parameter is μ .

Normal form:

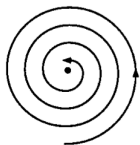
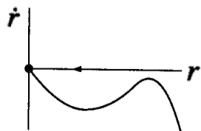
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



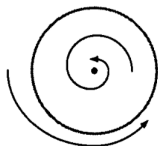
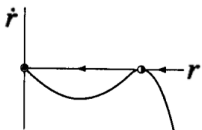
Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

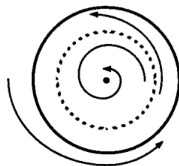
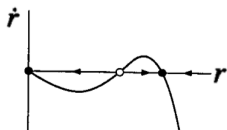
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \quad (7)$$



$$\mu < \mu_c$$



$$\mu = \mu_c$$

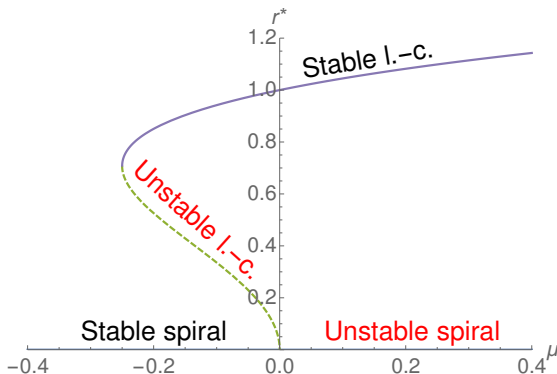


$$0 > \mu > \mu_c$$

Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

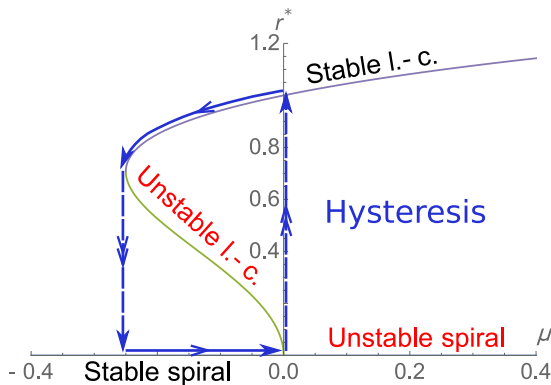
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Aeroelastic flutter



Credit: <https://www.youtube.com/watch?v=egDWh7jnNic>

The Tacoma Narrows bridge collapse, 1940



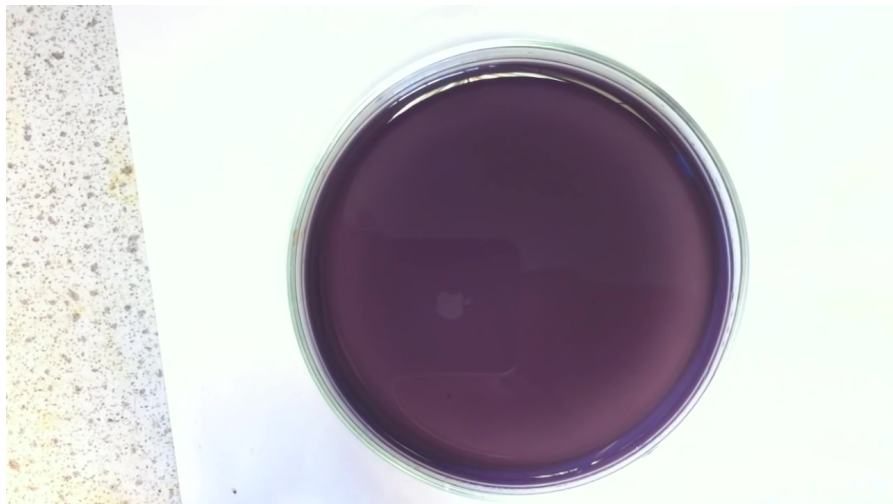
Credit: Barney Elliot, <https://www.youtube.com/watch?v=j-zczJXSxnw>

The Briggs–Rauscher oscillating reaction



Credit: Maxim Bilovitskiy, https://en.wikipedia.org/wiki/Briggs-Rauscher_reaction

The Belousov–Zhabotinsky reaction



Credit: Tim Kench, <https://www.youtube.com/user/timkenchh>

The tremor-dominant Parkinson's disease

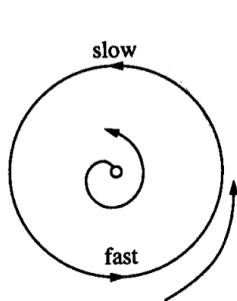


Credit: <https://doctorprodigious.wordpress.com>

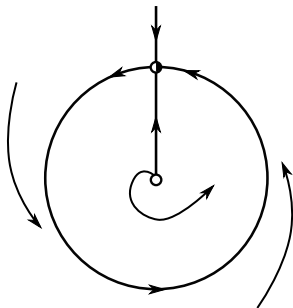
Case II: Bifurcations of closed orbits

Case II B: SNIPER or SNIC. Bif. parameter is ω . Example system:

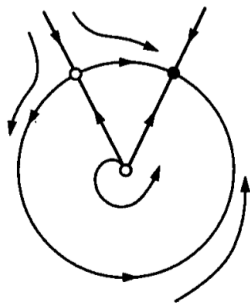
$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \omega - \sin \theta\end{aligned}\tag{8}$$



$$\omega > 1$$



$$\omega = 1$$



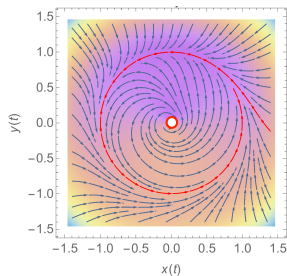
$$\omega < 1$$

Case II: Bifurcations of closed orbits

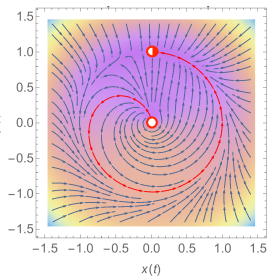
Case II B: SNIPER or SNIC. Bif. parameter is ω . Example system:

$$\dot{r} = r(1 - r^2)$$

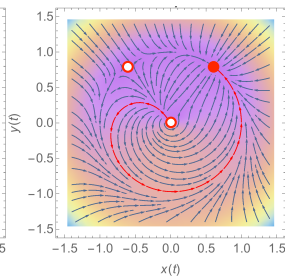
$$\dot{\theta} = \omega - \sin \theta$$



$\omega > 1$



$\omega = 1$

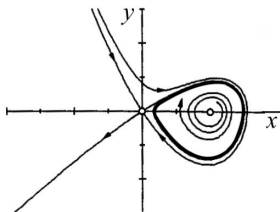


$\omega < 1$

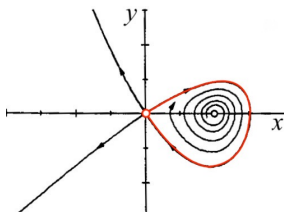
Case II: Bifurcations of closed orbits

Case II C: Homoclinic bifurcation. Bif. parameter is μ and $\mu_c \approx -0.8645$. Example system:

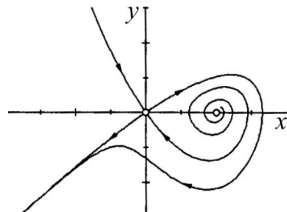
$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu y + x - x^2 + xy \end{cases} \quad (9)$$



$$\mu < \mu_c$$



$$\mu = \mu_c$$



$$\mu > \mu_c$$

Conclusions

- Classification of bifurcations in 2-D systems
- Bifurcations of fixed points
- Bifurcations of closed orbits
- The supercritical vs. subcritical Hopf bifurcations
- Dangers associated with the *Hopf* bifurcation
- Hysteresis on level of cycles

Revision questions

- Classification of bifurcations in 2-D.
- What is the Hopf bifurcation?
- What is the supercritical Hopf bifurcation?
- What is the subcritical Hopf bifurcation?
- What are global bifurcations of closed orbits?
- Name some global bifurcations of closed limit-cycles.
- What is a saddle-node coalescence (or bifurcation) of limit-cycles?
- What is hysteresis on level of cycles?
- Name dangers associated with the *Hopf* bifurcation.
- What is a saddle-node infinite period bifurcation?
- What is a (saddle-loop or) homoclinic bifurcation?
- Name examples of dynamical instabilities.