Lecture №6: 2-D conservative systems and centers, closed orbits and limit-cycles, the Dulac's criterion, the Poincaré-Bendixson theorem

Dmitri Kartofelev, PhD

Tallinn University of Technology,
School of Science, Department of Cybernetics, Laboratory of Solid Mechanics

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Lecture outline

- Conservative systems and centres
- Closed orbits and limit-cycles
- Importance of limit-cycles in applications
- How to detect closed orbits?
- Null-cline
- Heteroclinic orbit
- The Dulac's criterion
- The Poincaré-Bendixson theorem

Centers and conservative systems

Theorem: Suppose $\dot{\vec{x}} = \vec{f}(\vec{x})$ is conservative and $\vec{f}$ is continuously differentiable in $\vec{x} \in \mathbb{R}^2$. $E(\vec{x})$ is a conserved quantity and $\vec{x}^*$ is an **isolated fixed point**.

If that fixed point is a local minimum or maximum of $E(\vec{x})$, then that isolated fixed point $\vec{x}^*$ is a **center**, i.e., all trajectories close to $\vec{x}^*$ are closed orbits.

Mathematical pendulum

Mathematical pendulum[1] is given in the following normalised and dimensionless form:

$$\ddot{\theta} + \sin\theta = 0, \tag{1}$$

where θ is the angular displacement. For angular velocity $\omega = \dot{\theta}$ we rewrite the equation as a system of first order ordinary differential equations (ODEs):

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = -\sin\theta. \end{cases} \tag{2}$$

[1]See Mathematica .nb file uploaded to the course webpage.

Mathematical pendulum

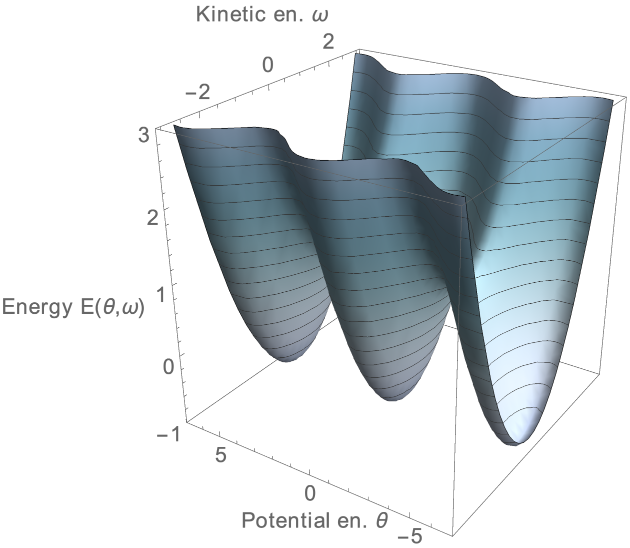


Figure: System Hamiltonian. Energy surface.

Mathematical pendulum

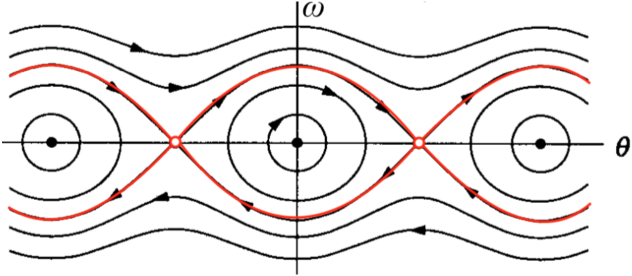


Figure: Phase portrait showing five fixed points $(\theta^*, \omega^*) = (-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$. Heteroclinic orbit is shown with the red coloured curves.

The Dulac's criterion

Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ be a continuously differentiable vector field defined on a *simply connected* subset R of a plane. If there exists a continuously differentiable, real valued function $g(\vec{x})$ such that

$$\mathrm{div}(g\dot{\vec{x}}) = \nabla\cdot(g\dot{\vec{x}}), \tag{3}$$

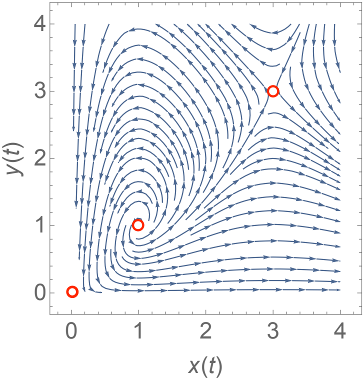
has one sign throughout R, then there are no closed orbits lying entirely in R.

Note: If the sign changes no conclusion can be drawn.

Example: The Dulac's criterion

Show that there are no closed orbits in region R for $x, y > 0$ if

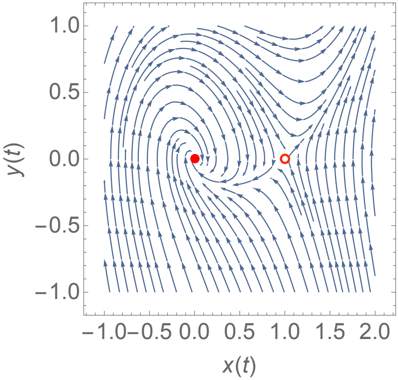
$$\begin{cases} \dot{x} = x(2 - x - y), \\ \dot{y} = y(4x - x^2 - 3). \end{cases} \tag{4}$$



Example: Dulac's criterion (*homework assignment*)

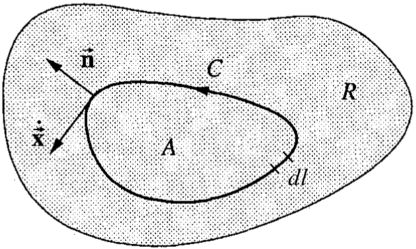
Show that there are no closed orbits in region $R \in \mathbb{R}^2$ for

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - y + x^2 + y^2. \end{cases} \tag{5}$$



Hint: Pick $g(\vec{x}) = \mathrm{e}^{-2x}$.

Proof by contradiction, the Dulac's criterion



Let C be a closed orbit in subset R, and let A be the region inside C. Green's theorem:

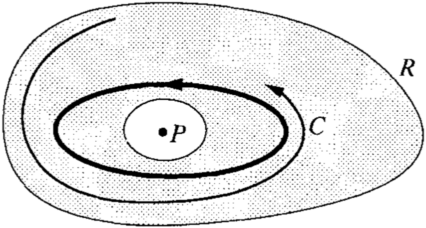
$$\boxed{\iint\limits_{A}\left(\nabla\cdot\vec{F}\right)\mathrm{d}A=\oint\limits_{C}\left(\vec{F}\cdot\vec{n}\right)\mathrm{d}l} \quad (6)$$

If $\vec{F}=g\dot{\vec{x}}$, then

$$\iint\limits_{A}\underbrace{\left[\nabla\cdot(g\dot{\vec{x}})\right]}_{\substack{\neq 0\\ \text{has one sign}\\ \text{by assumption}}}\mathrm{d}A=\oint\limits_{C}\underbrace{\left(g\dot{\vec{x}}\cdot\vec{n}\right)}_{\substack{=0\\ \vec{n}\perp\dot{\vec{x}}}}\mathrm{d}l \qquad ↯ \qquad (7)$$

Therefore there is no closed orbit C in R.

The Poincaré-Bendixson theorem



Suppose that:

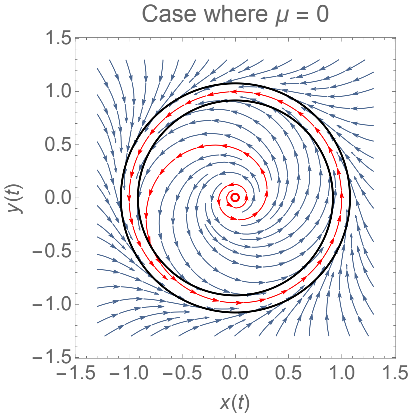
1. R is a closed, bounded subset in $\mathbb{R}^2$, called the **trapping region**;
2. $\dot{\vec{x}} = \vec{f}(\vec{x})$ is a continuously differentiable vector field on an open set containing R;
3. R does not contain any fixed points (P); and
4. there exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time.

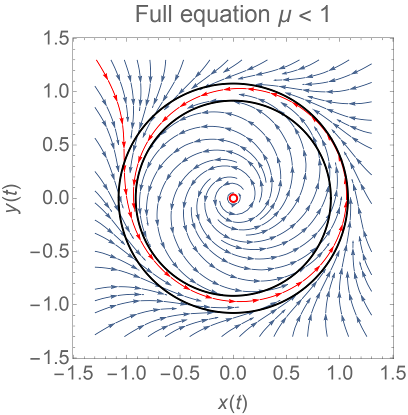
Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit, that is shown as a heavy closed curve in the above figure.

Example: The Poincaré-Bendixson theorem

Search for orbits in an annular region R for small μ given the system:

$$\begin{cases} \dot{r} = r(1-r^2) + \mu r\cos\theta, \\ \dot{\theta} = 1. \end{cases} \tag{8}$$





Glycolysis[2]: The Poincaré-Bendixson theorem

Search for closed orbits in an annular region R for glycolysis dynamics given by the following dimensionless and normalised system:

$$\begin{cases} \dot{x} = -x + ay + x^2 y, \\ \dot{y} = b - ay - x^2 y, \end{cases} \tag{9}$$

where a and b are the kinetic parameter groups, x and y are the concentrations of ADP and F6P molecules, respectively.

Read: Evgeni E. Sel'kov, "Self-oscillations in glycolysis 1. A simple kinetic model," *European Journal of Biochemistry*, **4**(1), pp. 79–86, (1968)

[2]See Mathematica .nb file uploaded to the course webpage.

Glycolysis, trapping region

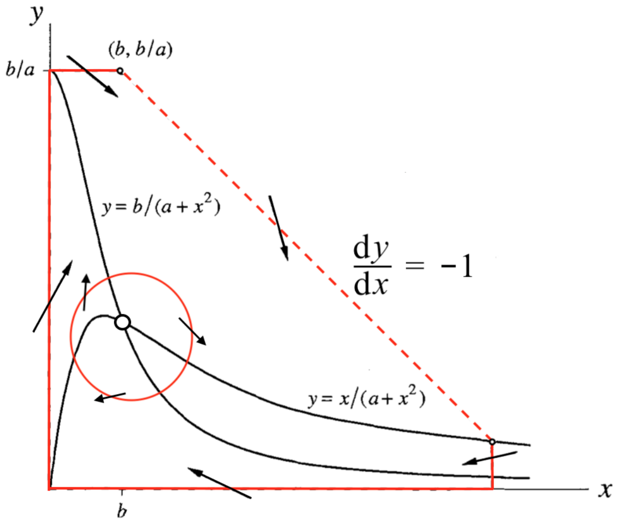


Figure: Annular trapping region shown with the red coloured lines and a circle. Local vector field flow directions are shown with the arrows.

Glycolysis, inner boundary of the trapping region

Secondly, we focus on the inner boundary of the proposed trapping region. We need to find and show that the fixed point:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{cases} -x^* + ay^* + x^{*2}y^* = 0 \\ b - ay^* - x^{*2}y^* = 0 \end{cases} \Rightarrow (x^*,\, y^*) = \left(b, \frac{b}{a+b^2}\right), \tag{10}$$

is unstable, i.e., the local vector field repels trajectories.

Glycolysis, inner boundary of the trapping region

We analyse fixed point (10) using linear analysis. The Jacobian of Sys. (9) has the following form:

$$J = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 2xy - 1 & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix}. \tag{11}$$

Glycolysis, inner boundary of the trapping region

The Jacobian evaluated at fixed point (10) takes the following form:

$$J\big|_{(x^*,y^*)} = \begin{pmatrix} \dfrac{2b^2}{a+b^2} - 1 & a+b^2 \\ -\dfrac{2b^2}{a+b^2} & -a-b^2 \end{pmatrix}. \tag{12}$$

It's determinant $\Delta = \det J\big|_{(x^*,y^*)} = a + b^2 > 0$ is positive because $a, b > 0$, and its trace

$$\tau = \operatorname{tr} J\big|_{(x^*,y^*)} = \frac{2b^2}{a+b^2} - 1 - a - b^2. \tag{13}$$

Glycolysis, inner boundary of the trapping region

In order to ensure repelling unstable fixed points for $\Delta > 0$, trace τ has to be positive. The dividing line between repelling unstable fixed points and stable ones is at $\tau = 0$. Solving

$$\tau = 0 \quad \Rightarrow \quad \frac{2b^2}{a+b^2} - 1 - a - b^2 = 0, \tag{14}$$

for parameter b gives

$$b(a) = \sqrt{\frac{1 - 2a \pm \sqrt{1-8a}}{2}}. \tag{15}$$

This result defines a line in the parameter space of Sys. (9). For parameter groups a and b in the region corresponding to $\tau > 0$, we are guaranteed that Sys. (9) has a closed orbit—an oscillating chemical reaction.

Glycolysis, inner boundary of the trapping region

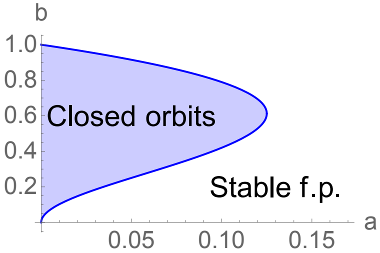
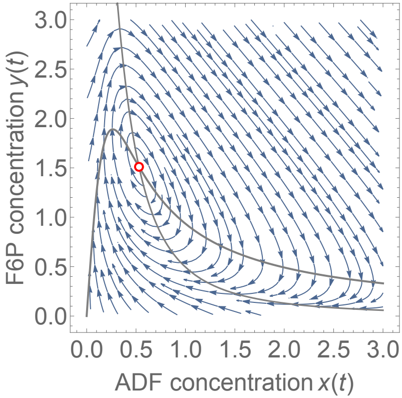
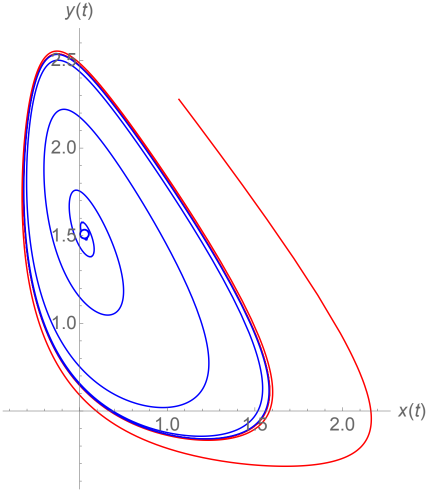


Figure: Parameter space defining the parameter values corresponding to the unstable fixed point given by (10).

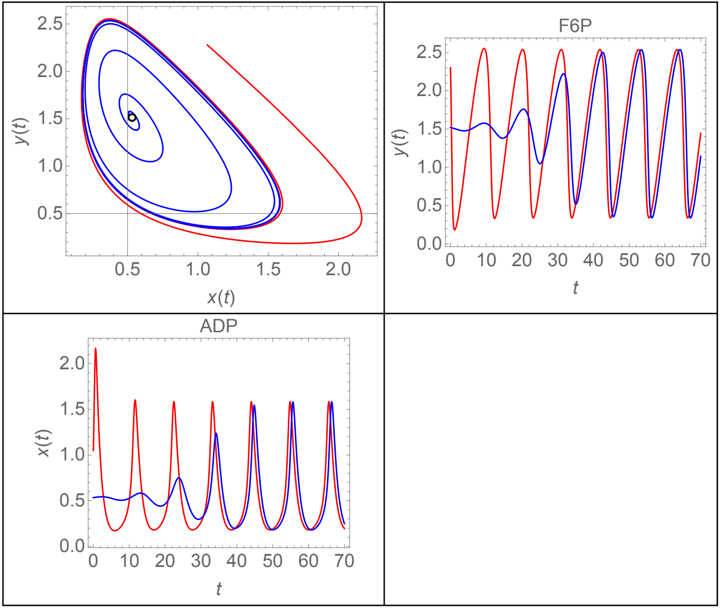
Glycolysis[3], limit-cycle





[3]See Mathematica .nb file uploaded to the course webpage.

Glycolysis, time-domain results



Conclusions

- Conservative systems
- Closed orbits and limit-cycles
- Importance of limit-cycles in applications
- Null-cline
- Heteroclinic orbit
- The Dulac's criterion
- The Poincaré-Bendixson theorem

Revision questions

- Expand on the connection between 2-D conservative systems and centers.
- Sketch a heteroclinic orbit.
- What is limit-cycle?
- Sketch a stable limit-cycle.
- Sketch an unstable limit-cycle.
- Sketch a half-stable (stable from outside) limit-cycle.
- Sketch a half-stable (stable from inside) limit-cycle.
- Define and sketch a null-cline.
- What is the Dulac's criterion?
- State the Poincaré-Bendixson theorem.
- Does the Poincaré-Bendixson theorem apply to 3-D systems?
- Can chaos occur in 2-D systems?