Lecture №5: 2-D homogeneous nonlinear systems, linearisation of 2-D systems about fixed points, stability of nonlinear fixed points, conservative systems

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Lecture outline

- Linearisation of 2-D systems about fixed points
- The Jacobian matrix of a system
- Stability analysis – linear fixed points vs. nonlinear fixed points
- Stable and unstable manifolds
- Nonlinear vs. linearised phase portrait
- Conservative systems
- Homoclinic orbit

Linearisation of 2-D systems

Nonlinear 2-D system for given functions f and g is defined by

$$\begin{cases} \dot{x} = f(x, y), \\ \dot{y} = g(x, y). \end{cases} \tag{1}$$

Let's consider two small perturbation: $|u| \ll 1$ in the x-direction, and $|v| \ll 1$ in the y-direction. The perturbed dynamics of the solution of Sys. (1) in close proximity to fixed point $(x^*,\, y^*)$ thus is

$$\begin{cases} x(t) = x^* + u(t), \\ y(t) = y^* + v(t), \end{cases} \tag{2}$$

equivalently we write:

$$\begin{cases} u(t) = x(t) - x^*, \\ v(t) = y(t) - y^*. \end{cases} \tag{3}$$

Linearisation of 2-D systems

Temporal dynamics of perturbations u and v is the following:

$$
\begin{cases}
\dot{u} = (x - x^*)^{\cdot} = \dot{x} = f(x^* + u, y^* + v) = \\
\quad = f(x^*, y^*) + u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*, y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*, y^*)} + O(u^2, v^2, uv) \approx \\
\quad \approx u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*, y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*, y^*)} \\
\\
\dot{v} = (y - y^*)^{\cdot} = \dot{y} = g(x^* + u, y^* + v) = \\
\quad = g(x^*, y^*) + u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*, y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*, y^*)} + O(u^2, v^2, uv) \approx \\
\quad \approx u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*, y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*, y^*)}
\end{cases}
\tag{4}
$$

Linearisation of 2-D systems

For a better overview we collect the above results:

$$\begin{cases} \dot{u} = u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,\, y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,\, y^*)}, \\[2ex] \dot{v} = u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,\, y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,\, y^*)}. \end{cases} \tag{5}$$

Linearisation of 2-D systems

The matrix form for $\vec{u} = (u, v)^T$ is the following:

$$\dot{\vec{u}} = \left.\begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}\right|_{(x^*,\, y^*)} \cdot \vec{u} \equiv J\big|_{(x^*,\, y^*)} \cdot \vec{u}, \qquad (6)$$

where matrix J is the Jacobian matrix of the given system. Neglecting higher order terms (h.o.t.) $O(u^2, v^2, uv)$ yields the linearisation about fixed point $(x^*,\, y^*)$ in form (6).

Note: Higher order terms of order $O(uv)$ are also negligibly small since $|u|, |v| \ll 1$.

Example: An ambiguous borderline case

An example where linear center is disturbed and changed by nonlinearity.

Consider the following system:

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2), \\ \dot{y} = x + ay(x^2 + y^2), \end{cases} \tag{7}$$

where a is the control parameter[1].

[1] See Mathematica .nb file uploaded to the course webpage.

Analysis of the nonlinear dynamics

Sys. (7) is analysed in polar coordinates. Usually a coordinate transform in the form:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \tag{8}$$

where $r = r(t)$ and $\theta = \theta(t)$, is used. This approach may prove to be work-intense. Let's instead use another valid identity in the form:

$$\boxed{\begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1} \dfrac{y}{x}. \end{cases}} \tag{9}$$

We are searching a system in the form:

$$\begin{cases} \dot{r} = f(r, \theta), \\ \dot{\theta} = g(r, \theta), \end{cases} \tag{10}$$

where functions $f(r, \theta)$ and $g(r, \theta)$ are to be determined.

Analysis of the nonlinear dynamics

Substituting (9) into original Sys. (7) results in

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) = -y + axr^2, \\ \dot{y} = x + ay(x^2 + y^2) = x + ayr^2. \end{cases} \tag{11}$$

Using (9) we write

$$r^2 = x^2 + y^2, \tag{12}$$

where $x = x(t)$, $y = y(t)$ and $r = r(t)$. We are interested in temporal dynamics, i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = \frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2), \tag{13}$$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} \quad | \div 2, \tag{14}$$

$$\boxed{r\dot{r} = x\dot{x} + y\dot{y}.} \tag{15}$$

This identity is used in connection with Sys. (11) to derive the first equation of sought Sys. (10).

Analysis of the nonlinear dynamics

Substituting (11) into the right-hand side of (15) results in

$$\begin{aligned} r\dot{r} &= x(-y + axr^2) + y(x + ayr^2) \\ &= -\cancel{xy} + ax^2r^2 + \cancel{xy} + ay^2r^2 \\ &= a\underbrace{\left(x^2 + y^2\right)}_{r^2}r^2 = ar^4. \end{aligned} \tag{16}$$

Above result can be simplified:

$$r\dot{r} = ar^4 \quad \big| \div r, \tag{17}$$

$$\boxed{\dot{r} = ar^3.} \tag{18}$$

We have found the first equation of sought Sys. (10). We are one step closer to the polar representation of the original problem, given by Sys. (7).

Analysis of the nonlinear dynamics

The second equation of sought Sys. (10) is found in the same way. Using (9) we study the temporal dynamics by writing:

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = \frac{\mathrm{d}}{\mathrm{d}t}\left(\tan^{-1}\frac{y}{x}\right) \Rightarrow \begin{bmatrix} \theta = \theta(t), \\ x = x(t), \\ y = y(t), \\ \text{chain rule,} \\ \text{simplify} \end{bmatrix} \Rightarrow 1 \cdot \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{\underbrace{x^2 + y^2}_{r^2}}. \tag{19}$$

Substituting (11) into the right-hand side of the obtained result gives:

$$\begin{aligned} \dot{\theta} &= \frac{x(x + ayr^2) - y(-y + axr^2)}{r^2} \\ &= \frac{x^2 + \cancel{axyr^2} + y^2 - \cancel{axyr^2}}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1, \end{aligned} \tag{20}$$

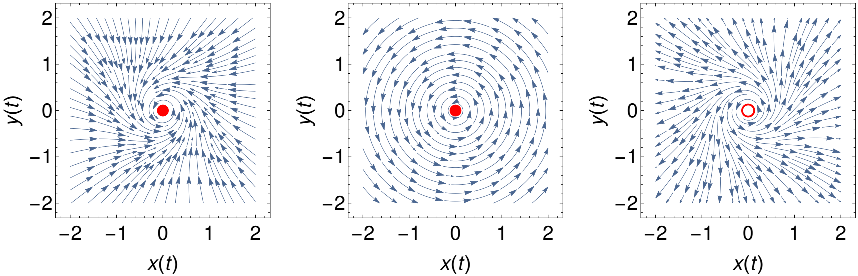
$$\boxed{\dot{\theta} = 1.} \tag{21}$$

We have found the second equation of sought Sys. (10).

Analysis of the nonlinear dynamics

Sys. (7) has been transformed into polar coordinates. Resulting decoupled equations (18) and (21) are the following:

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) \\ \dot{y} = x + ay(x^2 + y^2) \end{cases} \quad \Rightarrow \quad \begin{aligned} \dot{r} &= ar^3, \\ \dot{\theta} &= 1. \end{aligned}$$



$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$

Cohabitation model: Sheep and rabbits

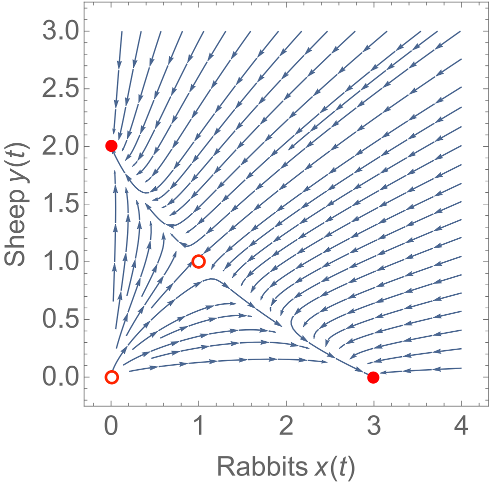
The Lotka-Volterra competitive cohabitation model[2] from ecology—competitive cohabitation of rabbits and sheep. The model has the following normalised and dimensionless form:

$$\begin{cases} \dot{x} = x(3-x) - 2xy, \\ \dot{y} = y(2-y) - xy, \end{cases} \tag{22}$$

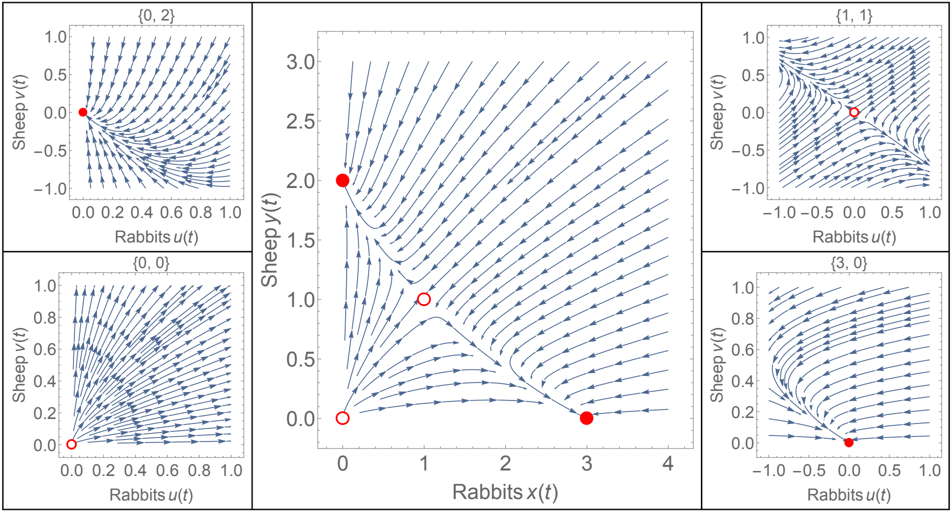
where x and y are the sizes of rabbit and sheep populations, respectively.

[2] See Mathematica .nb file uploaded to the course webpage.

Phase portrait of Sys. (22)



Phase portrait of Sys. (22), linear analysis



Phase portrait of Sys. (22)

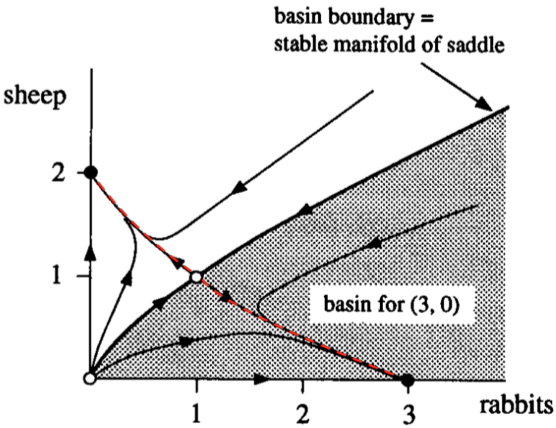


Figure: Phase portrait. The portrait features **two basins of attraction** corresponding to the stable fixed points. The **stable manifold of the saddle** is located at the basin boundaries. The unstable manifold is shown with the red dashed line.

Predator–prey model: Fish and sharks

Home assignment. Study the dynamics. How is this model different from the above presented "sheep and rabbits" model?

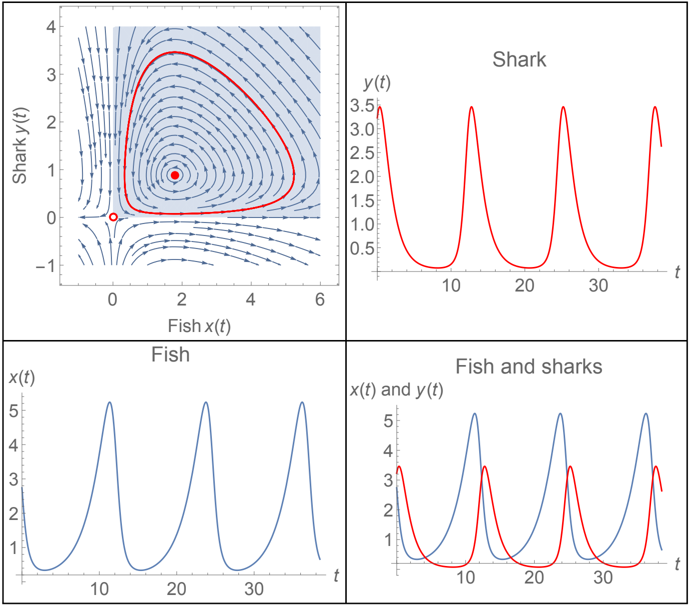
Model is given in the following normalised and dimensionless form:

$$\begin{cases} \dot{x} = \alpha x - \beta x y, \\ \dot{y} = \gamma \beta x y - \delta y, \end{cases} \tag{23}$$

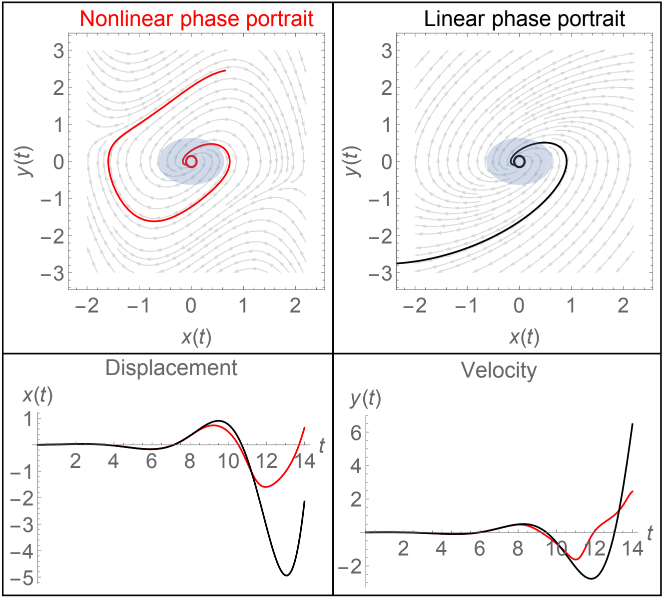
where x is the concentration of the prey species, y is the concentration of the predator species, α is the prey species' population growth rate, β is the predation rate of y upon x, γ is the assimilation efficiency of y, and δ is the mortality rate of the predator species[3].

[3] See Mathematica .nb file uploaded to the course webpage.

Predator–prey model: Fish and sharks



Nonlinear vs. linearised phase portrait



Example: The Liénard equation (red) and its linearisation (black). Parameter $\mu = 0.95$.

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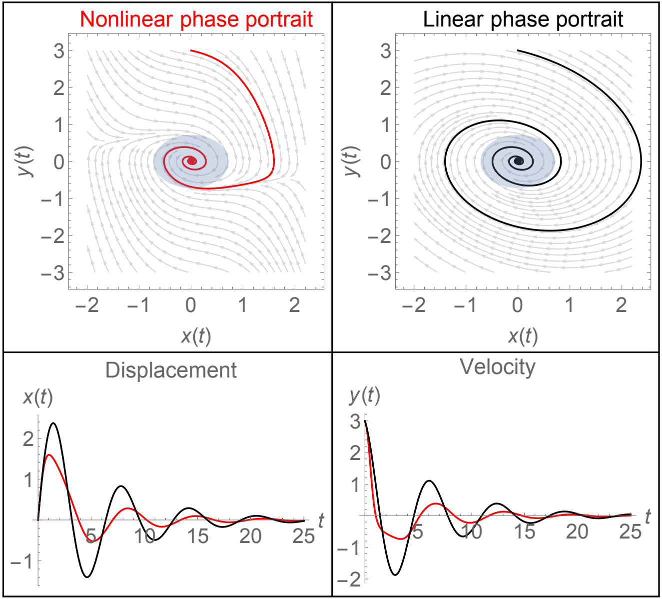
Nonlinear vs. linearised phase portrait



Example:
The Liénard equation (red) and its linearisation (black).
Parameter $\mu = -0.33$.

See Mathematica .nb file uploaded to the course webpage.

Conservative system

Consider a system with one degree of freedom given by an equation of motion in the form:

$$m\ddot{x} = F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}, \tag{24}$$

where m is the mass, V is the potential, and where force F is explicitly independent of time t (no external driving force) and $\dot{x}$ (no attenuation or damping terms).

In the conservative system the total energy is constant in time:

$$E = \frac{m\dot{x}^2}{2} + V(x) = \text{const.} \tag{25}$$

Conservative system, conserved quantity

Definition: Given a system $\dot{\vec{x}} = \vec{f}(\vec{x})$, a **conserved quantity** is a real-valued continuous function $E(\vec{x})$ that is constant on the system trajectories, i.e., $\mathrm{d}E/\mathrm{d}t = 0$.

To avoid trivial examples, we also require that $E(\vec{x})$ be non-constant on every open set. Otherwise a constant function like $E(\vec{x}) = 0$ would qualify as a conserved quantity for every system, and so every system would be conservative!

Conservative system

A proof from a classical mechanics textbook: Using Eq. (24) we write

$$m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = 0 \quad \Big| \cdot \dot{x}, \tag{26}$$

$$m\ddot{x}\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} = 0. \tag{27}$$

The left-hand side of (27) is a so-called **perfect derivative** or an **exact time-derivative**.

By applying the chain rule $\left(\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\right)$ in reverse we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{x}^2}{2} + V(x)\right) = 0, \tag{28}$$

from here it is clear that the sum of kinetic and potential energy do not change in time. Energy E is indeed a **conserved quantity**

$$\dot{E}(x, \dot{x}) = 0. \tag{29}$$

Conservative system: Example

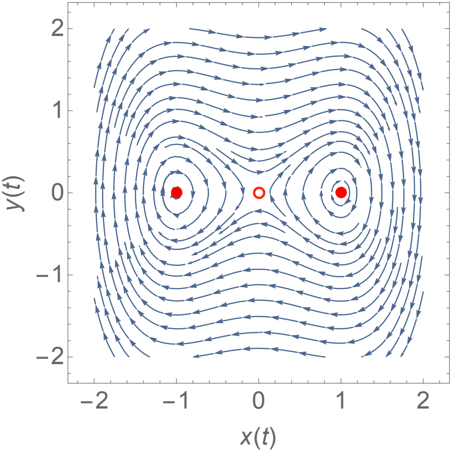
Example[4]: A particle in a double-well potential.

The potential energy is given by the following function:

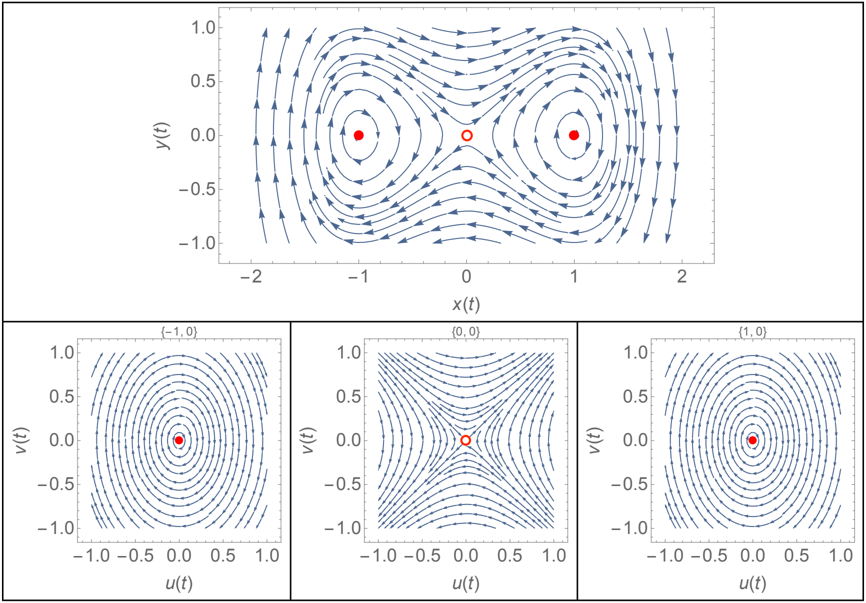
$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \tag{30}$$

[4]See Mathematica .nb file uploaded to the course webpage.

Particle in a double-well potential



Particle in a double-well potential, linear analysis



Particle in a double-well potential

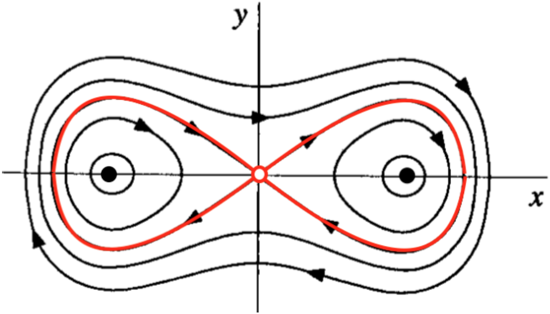
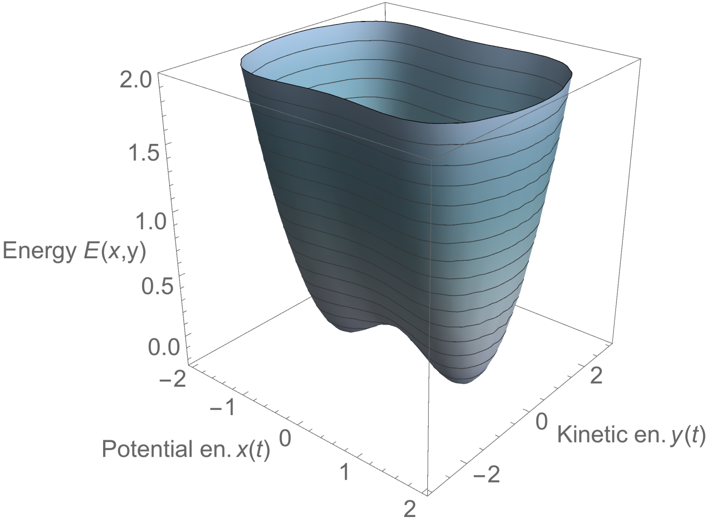


Figure: Phase portrait. The homoclinic orbit is shown with the red trajectories.

Particle in a double-well potential



Conclusions

- Linearisation of 2-D systems about fixed points
- The Jacobian matrix of a system
- Stability analysis – linear fixed points vs. nonlinear fixed points
- Stable and unstable manifolds
- Nonlinear vs. linearised phase portrait
- Conservative systems
- Homoclinic orbit

Revision questions

- Provide an example of nonlinear 2-D system.
- Explain linearisation of 2-D systems about fixed points.
- Can all nonlinear systems be linearised with the aim of identifying their fixed point type?
- Linearise the following system

$$\begin{cases} \dot{x} = 4x - 4xy, \\ \dot{y} = -9y + 18xy. \end{cases} \tag{31}$$

- Without taking derivatives, linearise the following systems:

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x, \end{cases} \tag{32}$$

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + y^2. \end{cases} \tag{33}$$

Revision questions

- Define Jacobian matrix of a system.
- Sketch a homoclinic orbit.
- Define conservative dynamical system.