

# Lecture №3: Over-damped bead on a rotating hoop, dimensionless form of equations of motion, introduction to 2-D systems, uniqueness of solution and phase space trajectories

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# Lecture outline

- Example from classical mechanics: Over-damped bead on a rotating hoop
- Dimensional analysis and dimensionless form of equations of motion
- 2-D systems or 2nd order systems
- 2-D phase portrait
- Consequences of existence and uniqueness of solutions of 2-D linear systems on phase portrait trajectories

## Ex: Over-damped bead on a rotating hoop

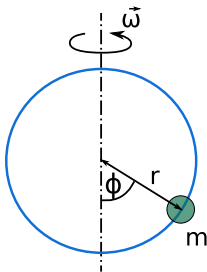


Figure: Scheme of the problem and used notations.

Derivation of the equation of motion in the following form:

$$mr\ddot{\phi} = -br\dot{\phi} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi). \quad (1)$$

## Ex: Over-damped bead on a rotating hoop

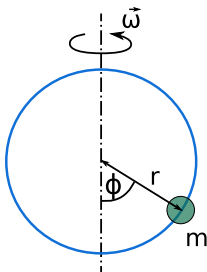
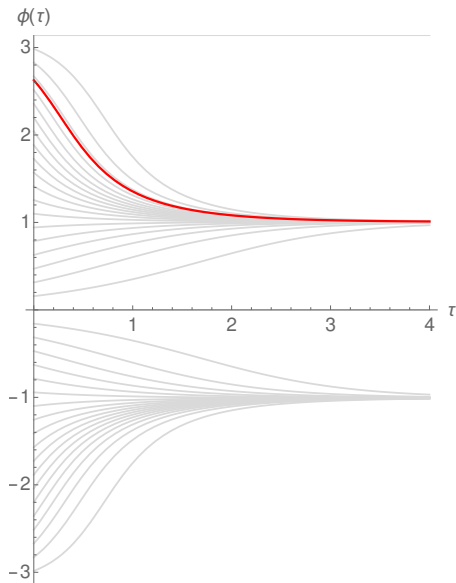


Figure: Scheme of the problem and used notations.

- Bifurcation analysis and construction of bifurcation diagram
- Dimensional analysis, non-dimensional normalised form

# Ex: Over-damped bead on a rotating hoop

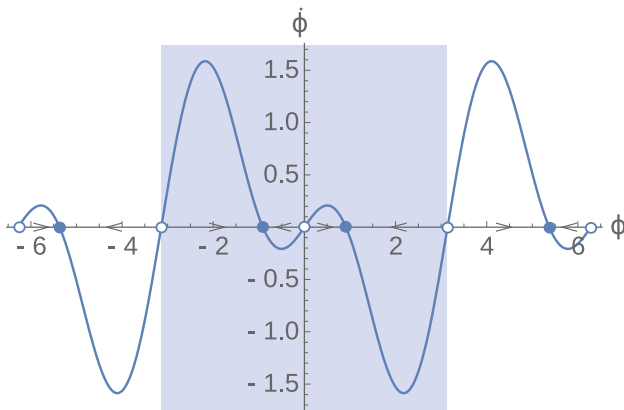


A family of solutions of 1-D problem in the form:

$$\frac{d\phi}{d\tau} = \sin(\phi) \left( \frac{r\omega^2}{g} \cos \phi - 1 \right)$$

# Ex: Over-damped bead on a rotating hoop

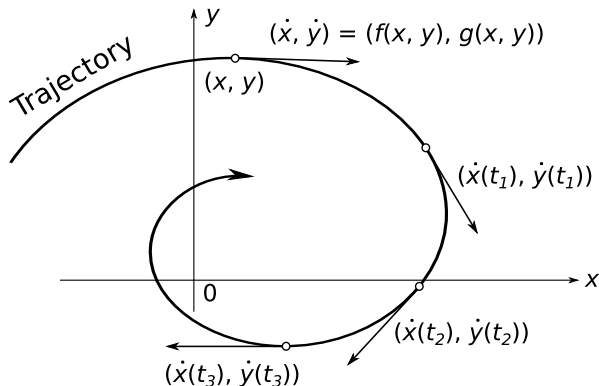
1-D approximation, phase portrait



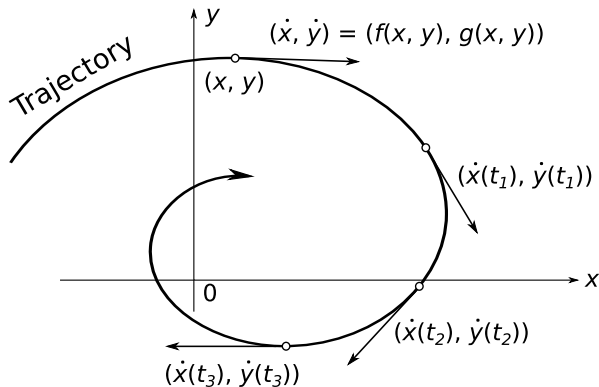
$$\frac{d\phi}{d\tau} = \sin(\phi) \left( \frac{r\omega^2}{g} \cos \phi - 1 \right) \quad (2)$$

## 2-D phase portrait

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \quad \text{or} \quad \dot{\vec{x}} = \vec{f}(\vec{x}) \quad (3)$$



## 2-D systems: Existence and uniqueness

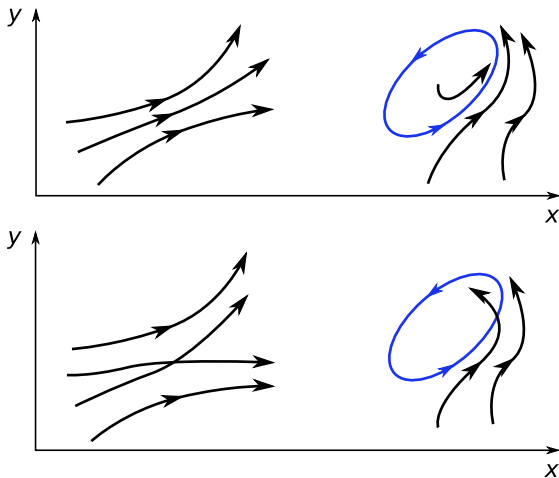


**Existence and uniqueness:** Solution to  $\dot{\vec{x}} = \vec{f}(\vec{x})$  exists and it is *unique* if  $\vec{f}(\vec{x})$  and  $\vec{f}'(\vec{x})$  are continuous. Function  $\vec{f}$  is continuously differentiable.



# 2-D systems

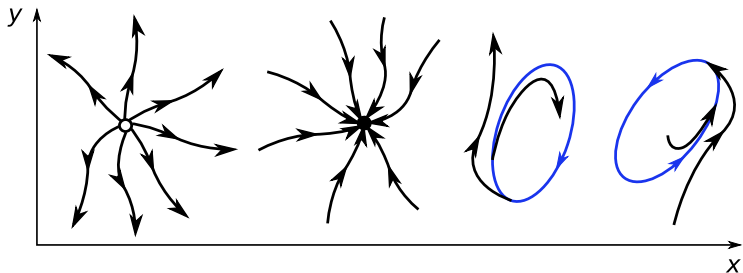
What does **uniqueness** and existence of solutions imply?



**Figure:** (Top) Possible dynamics. (Bottom) Impossible dynamics. The closed trajectory is showed with the blue closed curve.

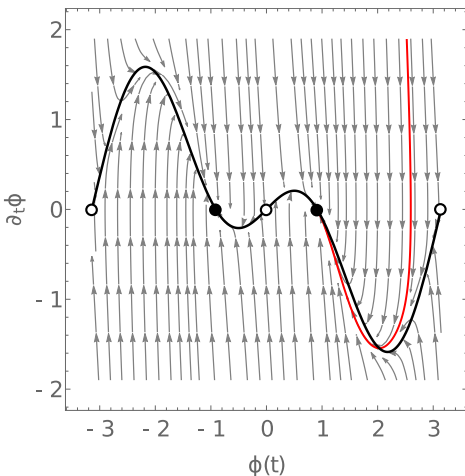
## 2-D systems

What does **uniqueness** and existence of solutions imply?



**Figure:** Additional possible dynamics: trajectories can approach or emanate from a fixed point because there is no motion at the fixed point, and they can merge with or emanate from closed trajectories (special type of dynamics called limit-cycles, only in nonlinear cases).

# Ex: Over-damped bead on a rotating hoop



Phase portrait of 2-D problem:

$$\alpha \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} + \sin(\phi)(\gamma \cos \phi - 1), \quad (4)$$

where

$$\alpha = \frac{m^2 g}{b^2 r}, \quad (5)$$

$$\gamma = \frac{r\omega^2}{g}. \quad (6)$$

1-D phase portrait of  $\frac{d\phi}{d\tau} = \sin(\phi)(\gamma \cos \phi - 1)$  is shown with black.

# Conclusions

- Example from classical mechanics: Over-damped bead on a rotating hoop
- Dimensional analysis and dimensionless form of equations of motion
- 2-D systems or 2nd order systems
- 2-D phase portrait
- Consequences of existence and uniqueness of solutions of 2-D linear systems on phase portrait trajectories

# Revision questions

- What is symmetry-broken solution?
- What is dimensional analysis of an equation of motion?
- What is dimensionless form of an equation?
- What is normalised form of an equation?
- How many initial conditions does first-order ODE have?
- How many initial conditions does second-order ODE have?
- Explain the notion of different time scales of a dynamical system.

# Revision questions

- Derive the dimensionless form of the following equation of motion:

$$m \frac{d^2 u}{dt^2} + b \frac{du}{dt} + ku = 0, \quad (7)$$

where  $u$  is the displacement,  $m$  is the mass and  $t$  is the time. Additionally, determine the dimensions of damping coefficient  $b$  and stiffness  $k$ .

- Derive the dimensionless form of the following nonhomogeneous equation of motion:

$$m \frac{d^2 u}{dt^2} + b \frac{du}{dt} + ku = F_0 \cos \omega_0 t, \quad (8)$$

where  $u$  is the displacement,  $m$  is the mass,  $F_0$  and  $\omega_0$  are the driving force parameters, and  $t$  is the time. Additionally, determine the dimensions of damping coefficient  $b$ , stiffness  $k$ , driving force  $F_0$  and driving force frequency  $\omega_0$ .