Lecture №15: Overview of the course, concluding remarks and summary, *exam*

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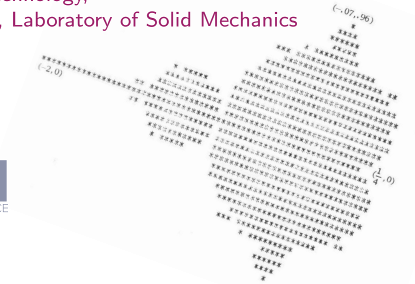


Outline

- Comments on your coursework evaluation results
- Overview of the course
- Highlights

- *Exam, what to expect?*

Intro to theory: basic definitions (from Lecture №1)

Mechanics (solid and fluid) can be roughly divided into **statics** and **dynamics**.

• **Statics:** the branch of mechanics concerned with the forces acting on stationary bodies. The acting forces are in equilibrium.

• **Dynamics:** the branch of mechanics concerned with the *motion*/changes of *bodies*/systems under the action of forces. The acting forces are not in equilibrium.
The branch of any science in which changes in variables are considered e.g. chemical kinetics, population biology, nonlinear oscillations, econophysics, etc. All these subjects can be placed under a common mathematical framework.

Nonlinear dynamics: concerns with dynamical systems or processes that are inherently *nonlinear*. Nonlinear dynamical systems, describing changes in variables over time, may appear *chaotic*, *unpredictable*, or *counterintuitive*, contrasting with much simpler linear systems.

Introduction, roadmap (from Lecture №1)

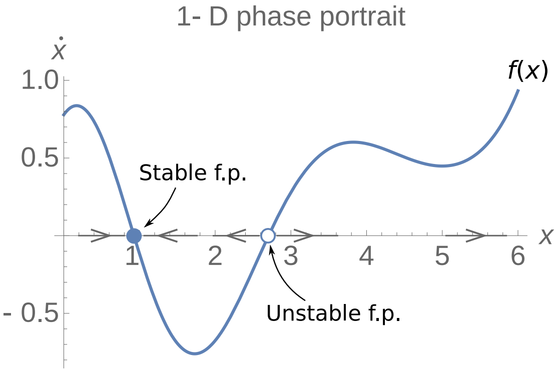
- 1-D systems (homogeneous ODEs)
- 2-D systems
  - Classification of fixed points (linear systems)
  - Classification of bifurcations
- *Quasi-periodisity.*
- 3-D systems and higher order systems
  - Strange attractors and chaos
  - Fractal geometry
  - Fractal microstructure of strange attractors
- *Poincaré map*
- 1-D maps and period doubling
- 2-D maps
  - Classification of fixed points (linear maps)
- Higher dimensional maps and complex valued maps

Overview of the course

The course can be divided into **six main milestones** and one overall conclusion:

1. Fundamentals of the course
2. Classification of fixed points and bifurcations
3. Periodicity, quasi-periodicity and aperiodicity
4. Chaos and three-dimensional systems
5. Analysis of chaos, long-term behaviour
6. Connection between nonlinear dynamical systems and the Mandelbrot set, and the Fatou and Julia sets

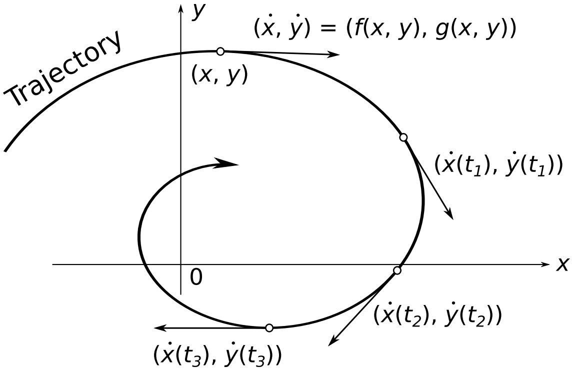
① Fundamentals of the course.



1-D flow problems and impossibility of oscillations in 1-D systems.

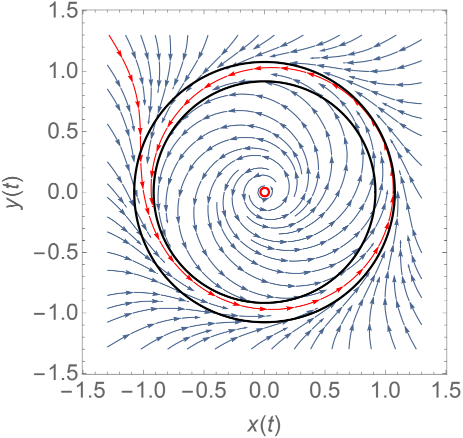
① Fundamentals of the course.

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$



2-D systems and 2-D phase portraits.

① Fundamentals of the course.



The Poincaré–Bendixson theorem and limit-cycles in continuous-time systems.

② Classification of fixed points and bifurcations

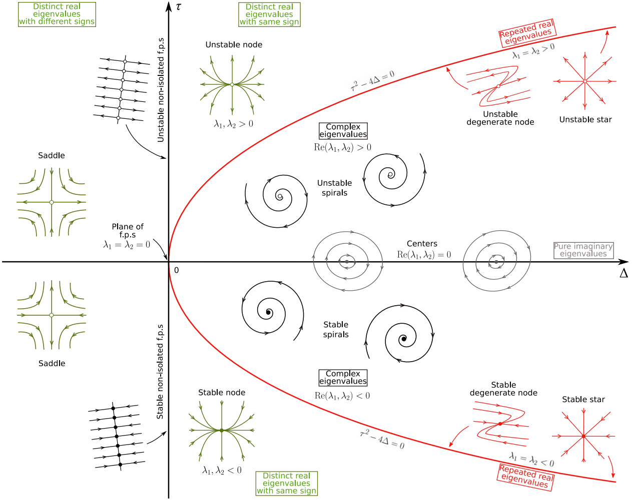


Figure: Classification of fixed points in 2-D linear systems.

② Classification of fixed points and bifurcations

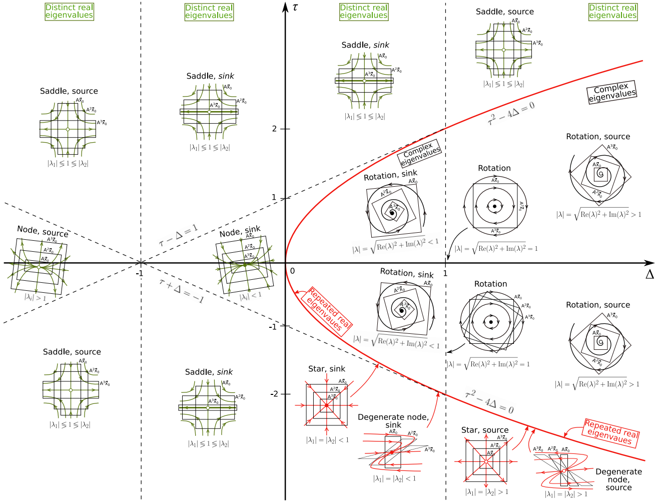


Figure: Classification of fixed points in 2-D linear maps.

Overview of the course

② Classification of fixed points and bifurcations

Case I Bifurcations of fixed points

- A) Bifurcations at $\lambda = 0$
  - 1) Saddle-node bifurcation
  - 2) Transcritical bifurcation
  - 3) Pitchfork bifurcation
    - * Supercritical pitchfork bifurcation
    - * Subcritical pitchfork bifurcation
- B) Hopf bifurcations, bifurcations at $\lambda = \pm i\omega$
  - 1) Supercritical Hopf bifurcation
  - 2) Subcritical Hopf bifurcation

Case II Bifurcations of closed orbits

- A) Coalescence of cycles (+ subcritical Hopf)
- B) SNIPER or SNIC
- C) Homoclinic bifurcation

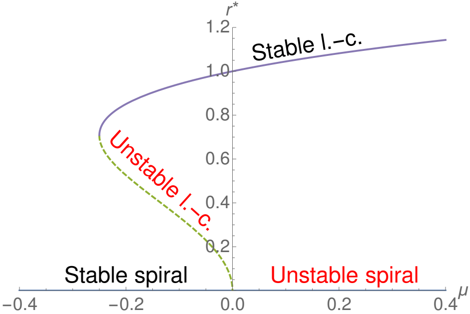


Figure: Classification of bifurcations and the *Hopf* bifurcation.

Overview of the course

② Classification of fixed points and bifurcations

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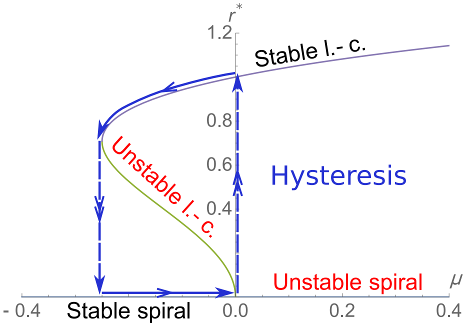


Figure: Classification of bifurcations and *Hopf* bifurcation.

Overview of the course

③ Periodicity, quasi-periodicity and aperiodicity

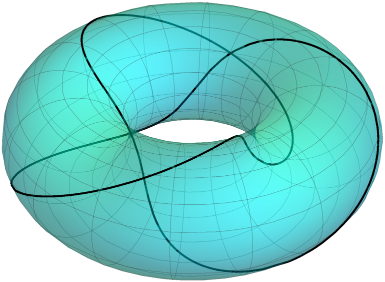
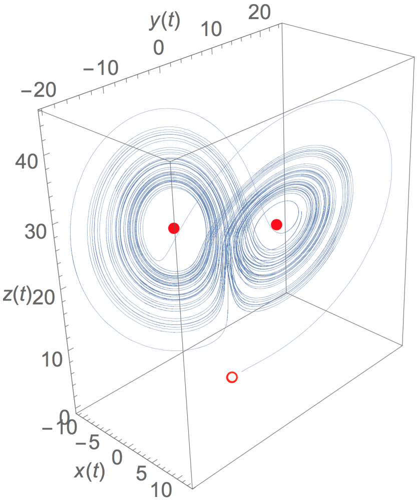


Figure: A closed trajectory on the surface of the torus. Toroidal knot.

Transitioning from 2-D to 3-D systems.

Overview of the course

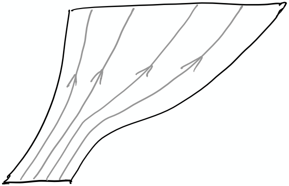
④ Chaos and 3-D systems (higher order systems)

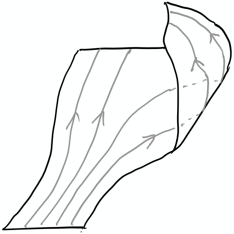


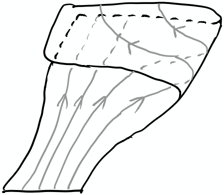
- Predictability horizon or the Lyapunov time
- The Lyapunov exponents
- The Kolmogorov entropy
- Chaotic or strange attractor

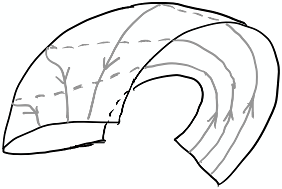
Overview of the course

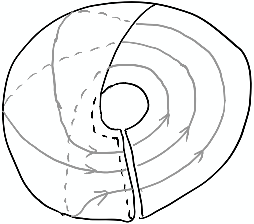
④ Chaos and 3-D systems (higher order systems)

1. 

2. 

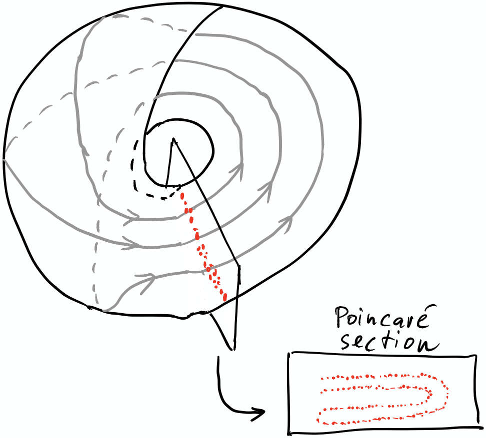
3. 

4. 

5. 

Overview of the course

④ Chaos and 3-D systems (higher order systems)



Number of layers in a real attractor $2^{\infty} = \infty$.

Fractal microstructure of strange attractors.
The stretching–folding–re-injection dynamics.

Overview of the course: Taffy pulling

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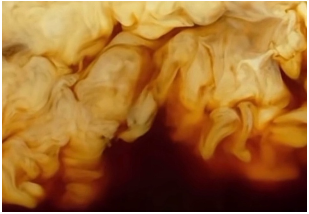
Overview of the course: Coffee and cream



Overview of the course: Air plume

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Overview of the course: Air

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Overview of the course: Ash cloud

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Overview of the course: Atmosphere

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Overview of the course

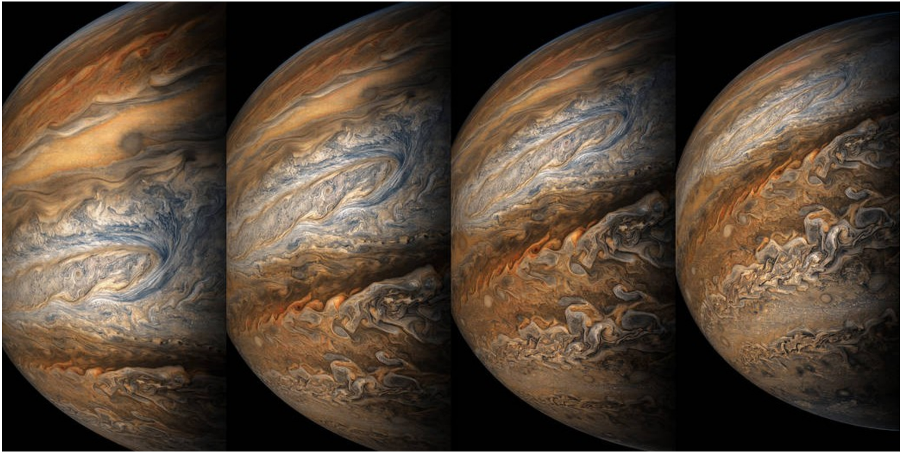


Figure: Jupiter. From left to right: the sequence of images taken on Sept. 1, 2017. At the time the images were taken, the spacecraft ranged from 12 143 to 22 908 km from the tops of the clouds of the planet.

Credit: NASA/JPL-Caltech/SwRI/MSSS/Gerald Eichstädt/Seán Doran

Overview of the course

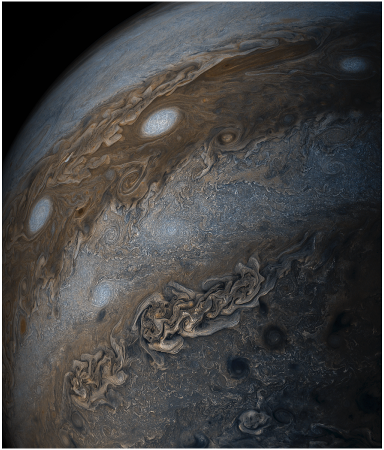


Figure: Left: Image acquired on May 19, 2017, from an altitude of about 33 400 km above Jupiter's cloud tops. Right: Image captured, from an altitude of 18 906 km.

Credit: NASA/JPL-Caltech/SwRI/MSSS/Gerald Eichstädt/Seán Doran

Overview of the course

Figure: Kalliroscope.

Overview of the course

④ Chaos and 3-D systems (higher order systems)

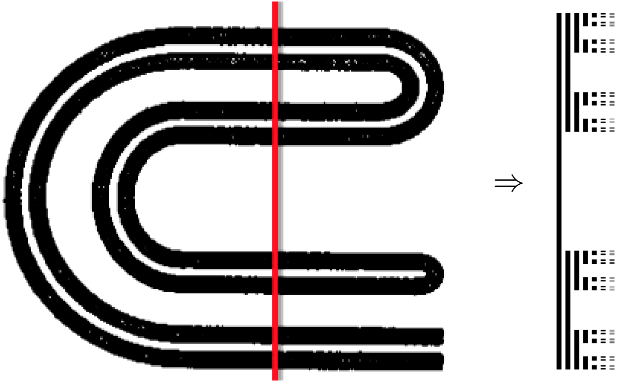
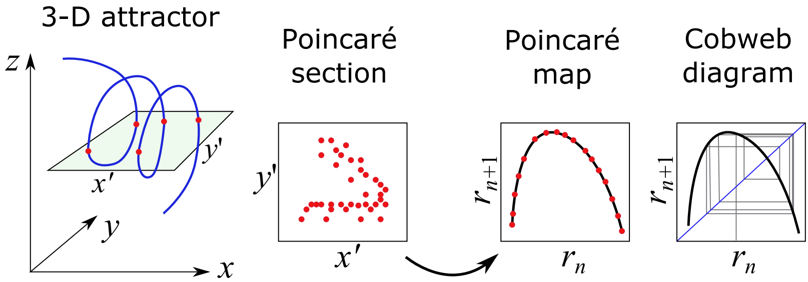


Figure: Local fractal microstructure of strange attractors. The Poincaré section and internal structure of the Lorenz section.

Overview of the course

⑤ Analysis of chaos, long-term behaviour. The Poincaré mapping of higher order systems.



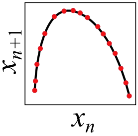
$$\begin{cases} x'_{n+1} = f_1(x'_n, y'_n) \\ y'_{n+1} = f_2(x'_n, y'_n) \end{cases} \Rightarrow \quad r_{n+1} = f_3(r_n) \tag{1}$$

Construction of the Poincaré map $\vec{P}(\vec{x}') = (f_1(x', y'), f_2(x', y'))^T$ (1). Mapping of the Poincaré section points where r is the radial distance from the origin ("flat" attractor).

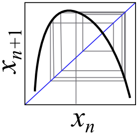
Overview of the course

⑤ Analysis of chaos, long-term behaviour.

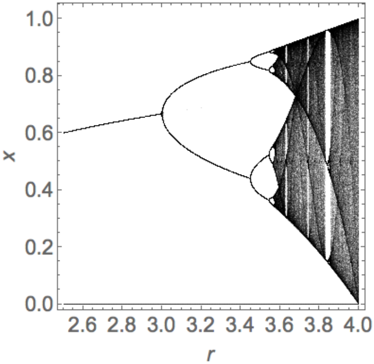
Poincaré map



Cobweb diagram

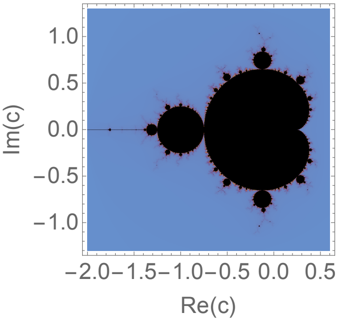


$\Rightarrow$



The Feigenbaum constants α, δ and the universal limiting functions – **universal route to chaos** via a cascade of period doubling bifurcations.

⑥ Connection between nonlinear dynamical systems and the Mandelbrot set, and the corresponding Fatou and Julia sets.



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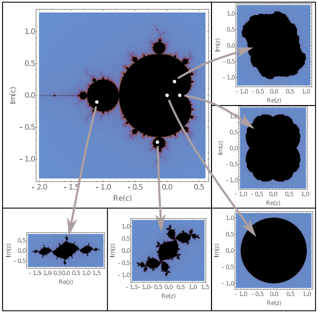


Parameter space and stable period-p cycles (polynomial space)

State spaces, initial conditions leading to all stable period-p orbits (basins of stable solutions)

Overview of the course, **main conclusions**

Be knowledgeable when dealing with nonlinearity!

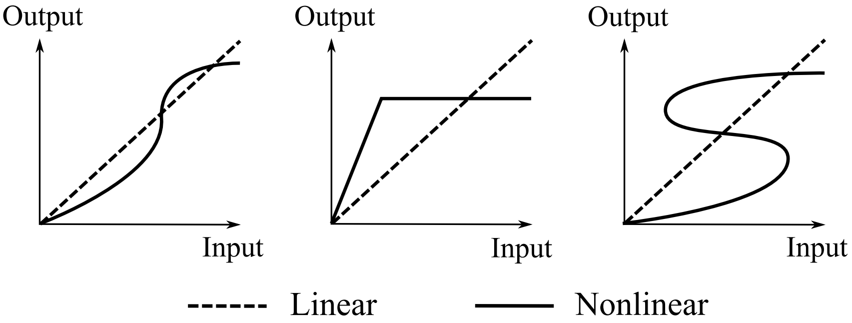


Figure: Nonlinear and linear systems.

Overview of the course, **main conclusions**

Be knowledgeable/careful when dealing with nonlinearity, especially when dealing with bifurcating nonlinear systems, systems in chaotic regime or systems with strange attractors. Keep in mind the predictability horizon and the Lyapunov exponents.

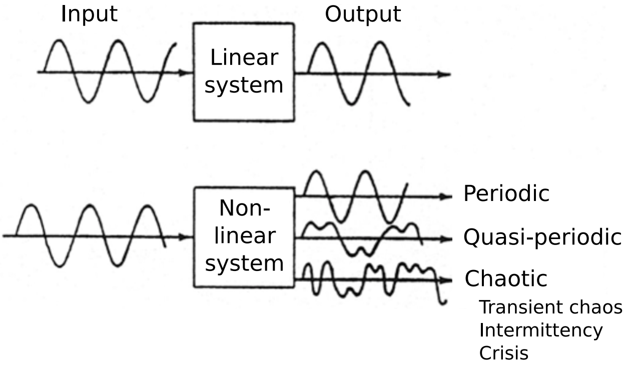
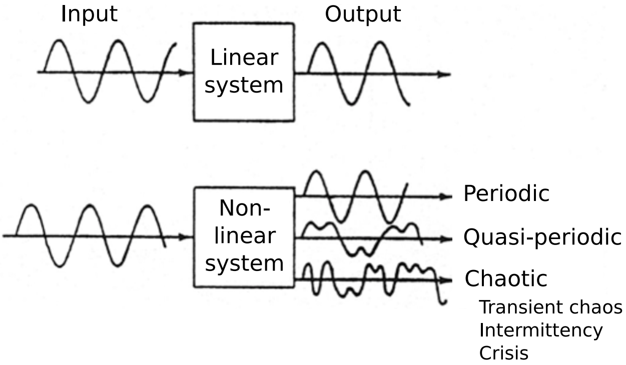


Figure: Comparison of linear and higher order nonlinear systems.

Overview of the course, **main conclusions**

Be knowledgeable/careful when dealing with nonlinearity, especially when dealing with bifurcating nonlinear systems, systems in chaotic regime or systems with strange attractors. Keep in mind the predictability horizon and the Lyapunov exponents.



Don't fuck with nonlinearity!

Exam, what to expect?

A practice mock exam is added to the lecture notes (Lecture 15) and the course webpage.

The final exam consist of:

- **Three** *simple* **questions or definitions** (12 points each) selected from the pool of the revision questions (non verbatim)
- **Three** *simple* **math problems** (for 12 or 13 points each)
- **Two video questions** (13 points each) designed to test your reasoning ability

Maximum points: 100. The exam is graded according to the TalTech grading scale.

Exam, an example video

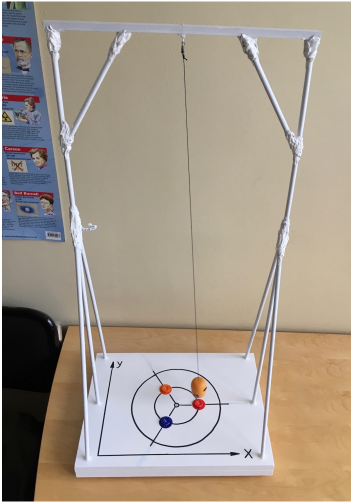


Figure: Magnetic pendulum in three magnetic potentials.

Exam, an example video

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Conclusions

- Comments on your coursework evaluation results
- Overview of the course
- Highlights

- *Next:* **Exam**

Reminder: The positively graded coursework is a prerequisite for taking the exam!