Lecture №12: Fractal microstructure of strange attractors, fractal geometry, fractal dimension, similarity and box dimensions, the Cantor set, the von Koch curve, 2-D maps, the Hénon map

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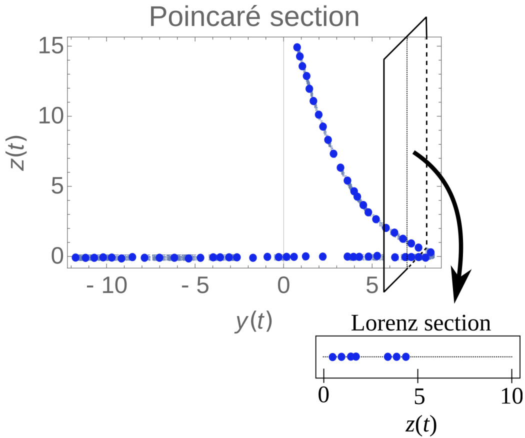
Lecture outline

- Geometry of strange attractors
- Analysis of stretch, fold and re-inject process in strange attractors
- Local microstructure of the Rössler attractor trajectories
- Introduction to fractal geometry: non-integer dimensions
- The Cantor set and von Koch curve (von Koch star)
- Similarity dimension
- Box counting dimension
- Introduction to 2-D maps
- The Hénon map as a simplified model of the Poincaré section of strange attractors
- Properties of the Hénon map

Geometry of strange attractors

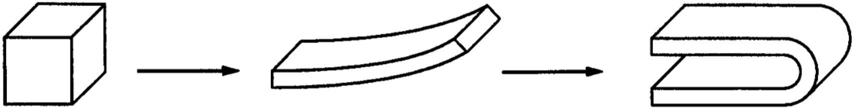
Microstructure of the Poincaré section of the Rössler attractor.



What happens inside the Lorenz section? What is the internal structure of the Lorenz section and how is it generated?

Stretching, folding and re-injecting

Cooking analogy — dough kneading

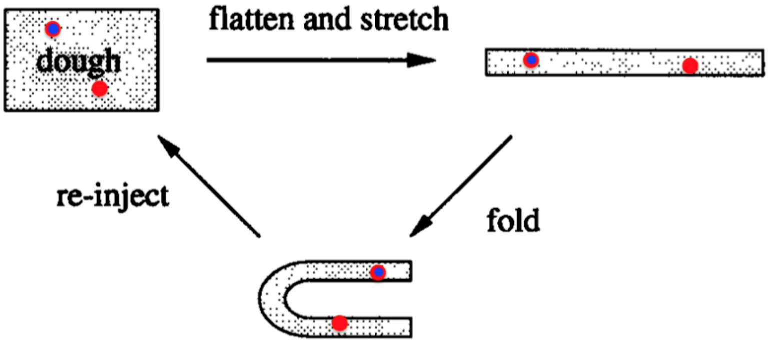


stretch

fold and repeat

Stretching, folding and re-injecting

Cooking analogy — dough kneading



Stretching, folding and re-injecting

After repeated re-injection steps.

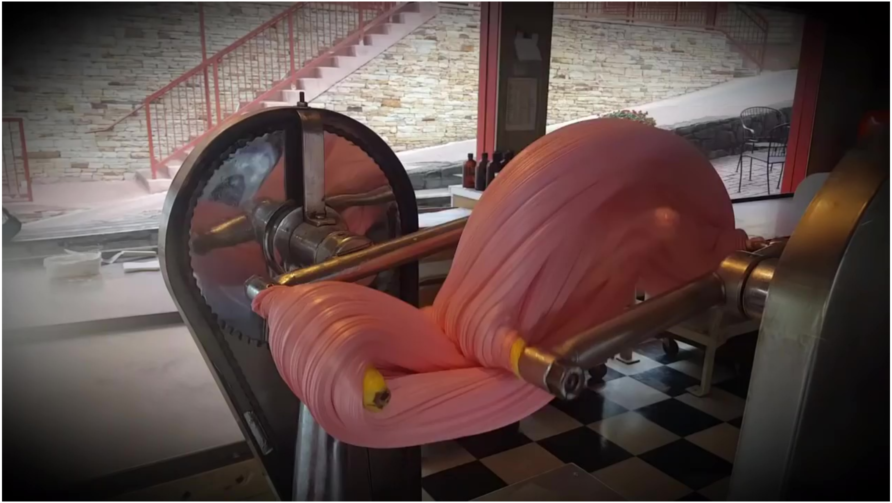
Resulting number of layers is 2^n, where n is the number of iterations.

Stretching, folding and re-injecting

Puff pastry dough (pâte feuill)

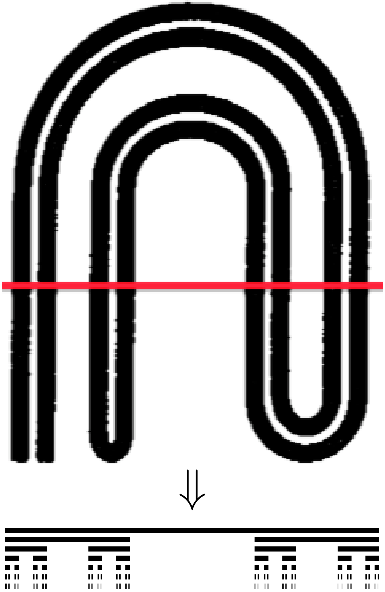
Resulting puff pastry (pâte feuilletée)

Taffy pulling



Credit: Cubbies Fan #1, https://www.youtube.com/watch?v=XnndSkcj1Bw

The Lorenz section



The Poincaré section

The Lorenz section

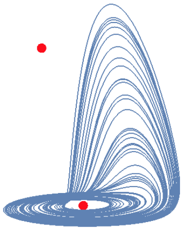
Internal structure of the Lorenz section

The Rössler attractor

The Rössler attractor[1] has the form (as mentioned in Lectures 9, 11):

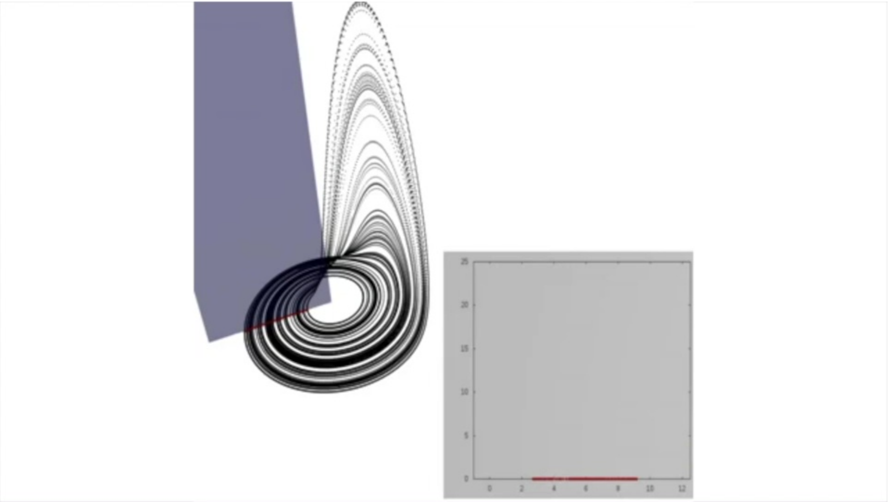
$$\begin{cases} \dot{x} = -y - x, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases} \tag{1}$$

Chaotic solution exists for $a = 0.1$, $b = 0.1$, $c = 14$.



[1]See Mathematica .nb file uploaded to the course webpage.

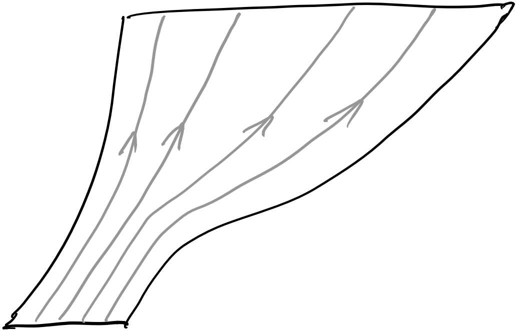
Poincaré section dynamics in the Rössler attractor



Credit: Timothy Jones, `http://www.physics.drexel.edu/~tim/`

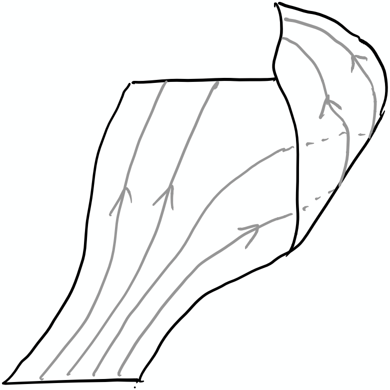
Geometry of the Rössler attractor

1.



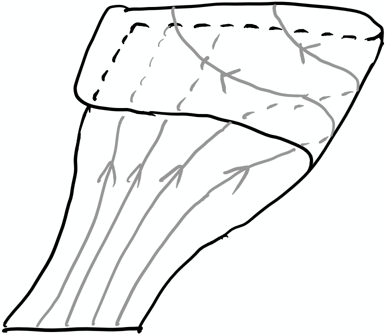
Geometry of the Rössler attractor

2.



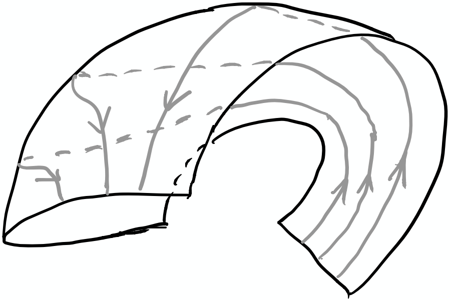
Geometry of the Rössler attractor

3.



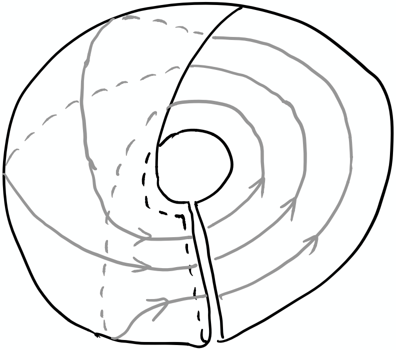
Geometry of the Rössler attractor

4.



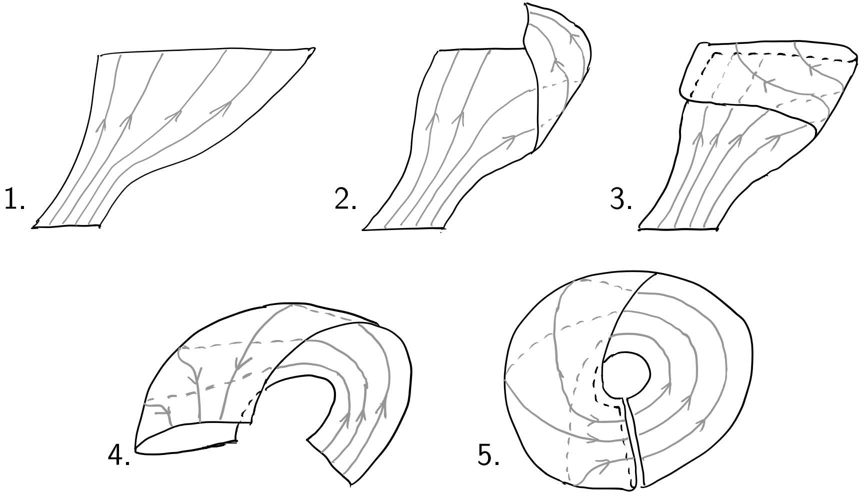
Geometry of the Rössler attractor

5.



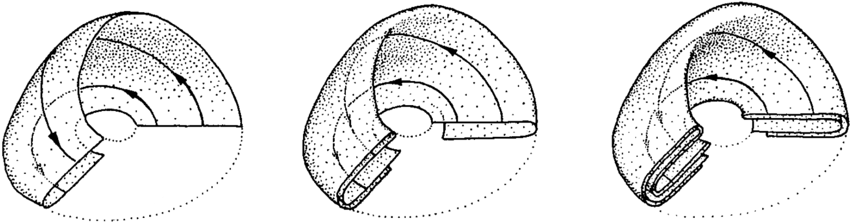
Do not forget about the uniqueness of solutions.

Geometry of the Rössler attractor[2]



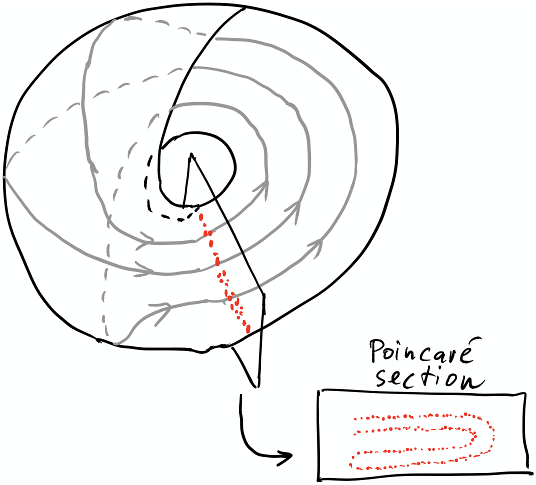
[2]See Mathematica .nb file uploaded to the course webpage.

Geometry of the Rössler attractor



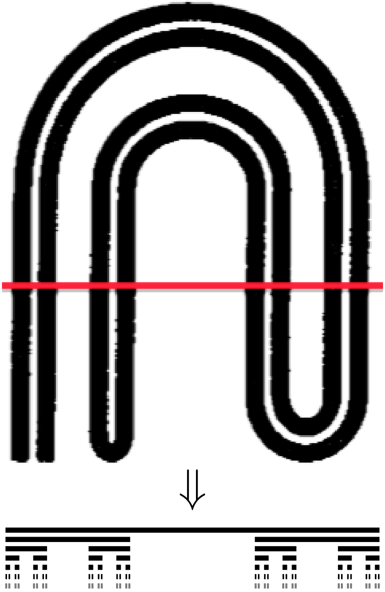
Credit: Abraham & Shaw, "Dynamics, the Geometry of Behavior, Part 2: Chaotic Behavior," 1983, pp. 121–123, reproduced in Strogatz, "Nonlinear Dynamics and Chaos," 1994 edition, p. 436

Geometry of the Rössler attractor



Number of layers in a real attractor $2^{\infty} = \infty$.

The Cantor set and the Lorenz section

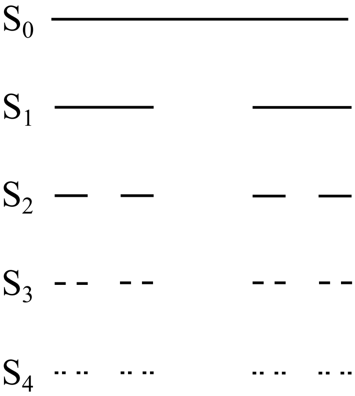


The Poincaré section

The Lorenz section

Internal **infinite** structure of the Lorenz section, 2^{∞} layers

The Cantor set[3] shown to the fourth iterate (pre-fractal).



$$S_0 = 1$$

$$S_1 = \frac{2}{3}$$

$$S_2 = \left(\frac{2}{3}\right)^2$$

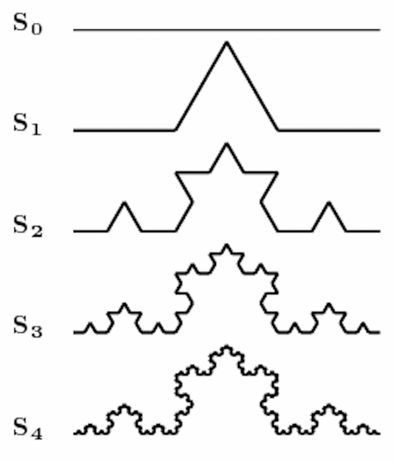
$$S_3 = \left(\frac{2}{3}\right)^3$$

$$S_4 = \left(\frac{2}{3}\right)^4$$

[3] See Mathematica .nb file uploaded to the course webpage.

Properties of the von Koch curve

The von Koch curve[4] shown to the fourth iterate.



$$S_0 = 1$$

$$S_1 = \frac{4}{3}$$

$$S_2 = \left(\frac{4}{3}\right)^2$$

$$S_3 = \left(\frac{4}{3}\right)^3$$

$$S_4 = \left(\frac{4}{3}\right)^4$$

[4] See Mathematica .nb file uploaded to the course webpage.

Properties of the von Koch curve

Selected properties of the von Koch curve:

- Measure (total length) of the curve is ∞

$$|S_0| = 1, \quad |S_1| = \frac{4}{3}, \quad |S_2| = \left(\frac{4}{3}\right)^2, \quad |S_3| = \left(\frac{4}{3}\right)^3, \ldots$$

$$|S_n| = \left(\frac{4}{3}\right)^n, \quad \text{where} \quad n \in \mathbb{Z}^+ \tag{2}$$

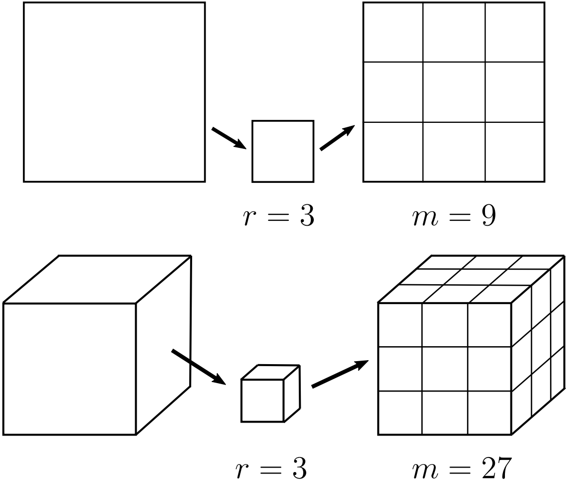
$$|S_\infty| = \lim_{n\to\infty} |S_n| = \infty \tag{3}$$

- Distance between any two points is ∞, see (3)
- Self-similar — the curve is comprised of smaller copies (reduced by factor of 3) of itself
- Non-integer similarity and box dimension:

$$d = \frac{\ln 4}{\ln 3} \approx 1.26 \tag{4}$$

A way to think about dimensionality

Idea behind the similarity dimension.



Rule: $m = r^2$

Rule: $m = r^3$

The power of r carries information about the dimensionality.

Similarity dimension

If one can demonstrate the existence of a scaling relation in the form

$$m = r^d \quad \Rightarrow \quad \boxed{d = \frac{\ln m}{\ln r},} \tag{5}$$

where r is the reduction factor of the self-similar substructures and m is the number of the self-similar substructures that comprise the original object.

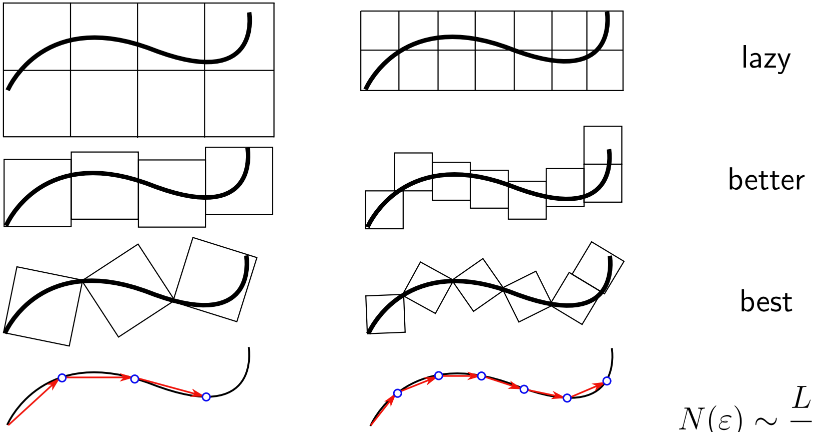
Then we have d — a **similarity dimension**.

Example: The Cantor set, a scaling law for iterates of the fractal exist and one can find that

$$d = \frac{\ln m}{\ln r} = \frac{\ln 2}{\ln 3} \approx 0.63.$$

Box counting dimension

Generalisation of the similarity dimension. Below: L is the line length, ε is the box size and N is the number of covering boxes.

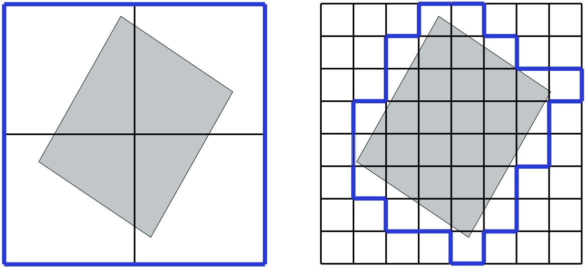


$\varepsilon = \varepsilon_1$ $\varepsilon = \varepsilon_2 < \varepsilon_1$

$$N(\varepsilon) \sim \frac{L}{\varepsilon} \quad (6)$$

Box counting dimension in 2-D

A is the area of the rectangle, ε is the box size, and N is the number of covering boxes.



$\varepsilon = \varepsilon_1$ $\qquad$ $\varepsilon = \varepsilon_2 < \varepsilon_1$

$$N(\varepsilon) \sim \frac{A}{\varepsilon^2} \qquad (7)$$

The scaling law also holds for the Euclidean D-dimensional subset regions R where $D > 2$ (the covering Euclidean cubes are D-dimensional).

$$N(\varepsilon) \sim \frac{R}{\varepsilon^d} \qquad (8)$$

Box counting dimension, power law

$$N(\varepsilon) \sim \frac{R}{\varepsilon^d} \quad \Rightarrow \quad N(\varepsilon) \sim \frac{1}{\varepsilon^d}, \tag{9}$$

where R is the (unit) volume. Solving for the power d gives

$$N(\varepsilon) \sim \frac{1}{\varepsilon^d} \quad \Big| \cdot \varepsilon^d, \tag{10}$$

$$\varepsilon^d N(\varepsilon) \sim 1 \quad \Big| \ln(\cdot), \tag{11}$$

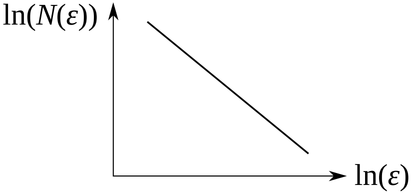
$$d \ln \varepsilon + \ln N(\varepsilon) \sim 0, \tag{12}$$

$$d \sim -\frac{\ln N(\varepsilon)}{\ln \varepsilon} \sim \left[\ln(1/\varepsilon) \overset{\text{since}}{=} \underbrace{\ln 1}_{=0} - \ln \varepsilon \right] \sim \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}. \tag{13}$$

Box counting dimension, power law

$$N(\varepsilon) \sim \frac{1}{\varepsilon^d}$$

Conclusion: the power of ε carries information about the dimensionality of an object.



Slope of the log-log plot

$$\boxed{d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}.} \qquad (14)$$

If limit $\varepsilon \to 0$ of d exist we have a **box counting dimension** (a power law).

This approach also allows to estimate non-Euclidean, non-integer dimensions of fractals and fractal-like objects.

2-D maps: the Hénon map

Hénon created a simple fractal attractor in order to study the general properties of strange attractors.

The 2-D Hénon map[5] has the following form:

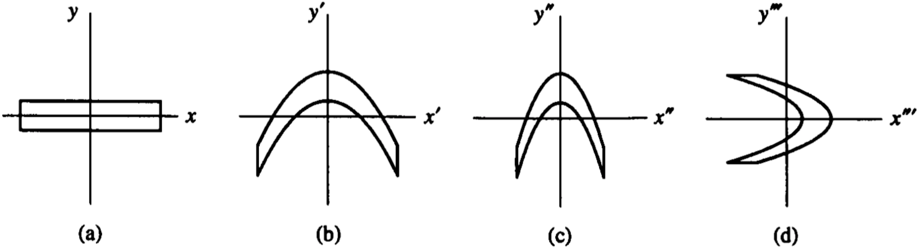
$$\begin{cases} x_{n+1} = y_n + 1 - a x_n^2, \\ y_{n+1} = b x_n, \end{cases} \tag{15}$$

where a and b are the control parameters. Chaotic solution exists for $a = 1.4$, $b = 0.3$.

Read: M. Hénon, "A two-dimensional mapping with a strange attractor," *Communications in Mathematical Physics*, Vol. 50, No. 1 (1976), pp. 69–77.

[5] See Mathematica .nb file uploaded to the course webpage.

Modelling the stretch–fold–re-inject dynamics of the Poincaré section of a 3-D attractor:



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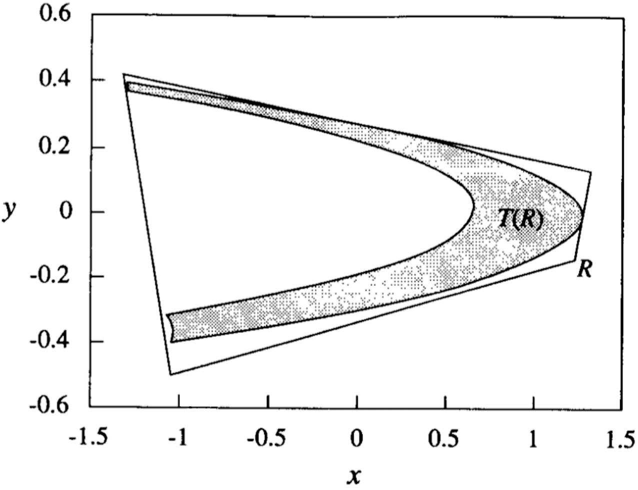
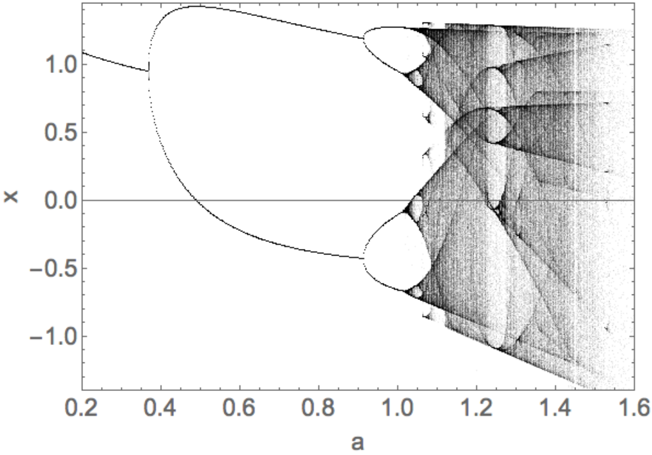
The Hénon map, local trapping region R



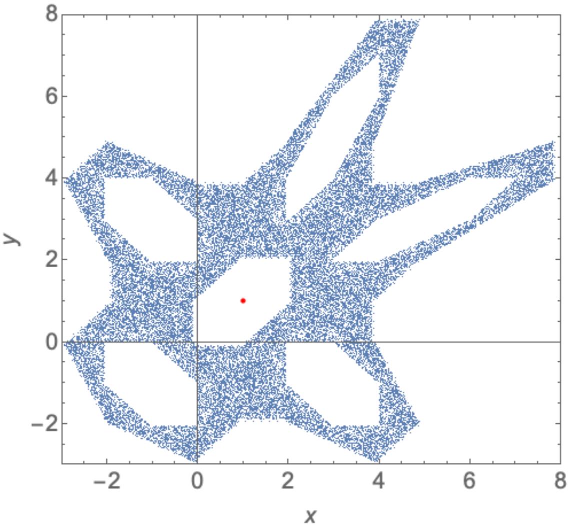
The Hénon map, orbit diagram[6]



Period doubling and chaotic dynamics are present for $b = 0.3$.

[6] See Mathematica .nb file uploaded to the course webpage.

Gingerbread man map or fractal[7]



Showing $4 \cdot 10^4$ iterates for initial conditions $x_0 = -0.3$ and $y_0 = 0$.

[7]See Mathematica .nb file uploaded to the course webpage.

Conclusions

- Geometry of strange attractors
- Analysis of stretch, fold and re-inject process in strange attractors
- Local microstructure of the Rössler attractor trajectories
- Introduction to fractal geometry: non-integer dimensions
- The Cantor set and von Koch curve (von Koch star)
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Revision questions

- How is it possible for two trajectories with almost equal initial conditions to deviate exponentially and remain attracted to a strange attractor (remain in the basin of attraction)?
- Give an example of a dynamics that features global stability and local instability.
- Explain fractal microstructure of strange attractors.
- Define fractal.
- What is pre-fractal?
- Construct a simple fractal (general idea).
- What is self-similarity?
- What is scale-invariance?
- Are all fractals self-similar?
- What is fractal geometry?

Revision questions

- What is fractal dimension?
- What are similarity and box counting dimensions?
- What is a power law?
- What is the Cantor set?
- What is the von Koch curve?
- What is a 2-D map?
- How to find fixed points of 2-D maps (period-1 point)?
- What is the Hénon map, its significance?