Lecture №11: Feigenbaum's analysis of period doubling, renormalisation, universal limiting function, discrete-time dynamics analysis, the Poincaré section, the Poincaré map, the Lorenz section, attractor reconstruction

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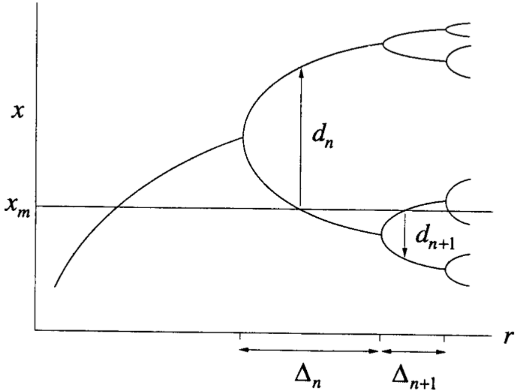
Lecture outline

- Feigenbaum's analysis of period doubling
- The universal route to chaos
- Universal aspects of period doubling in unimodal maps
- Superstable fixed points and period-p orbits
- Renormalisation
- Universal limiting functions and the onset of chaos
- Discrete-time dynamics analysis methods
  - The Poincaré section
  - Return map or the Poincaré map
  - The Lorenz section
- Non-homogenous systems
- Examples studied:
  - The periodically forced damped Duffing oscillator
  - The Rössler attractor

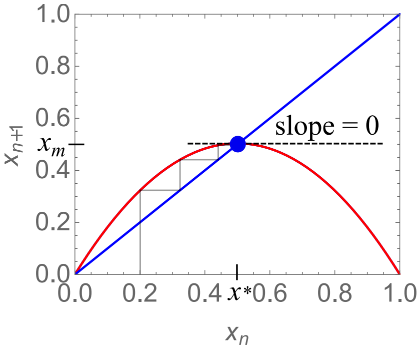
1-D unimodal maps and the Feigenbaum constants



$$\delta = \lim_{n\to\infty} \frac{\Delta_{n-1}}{\Delta_n} = \lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609\ldots \quad (1)$$

$$\boxed{\alpha = \lim_{n\to\infty} \frac{d_{n-1}}{d_n} \approx -2.502907875\ldots} \quad (2)$$

Superstable fixed point (the logistic map)



$$f(x^*, r) = x^*, \quad x^* = x_m, \quad \text{and} \quad f'(x^*, r) = 0 \quad \Rightarrow r = 2. \qquad (3)$$

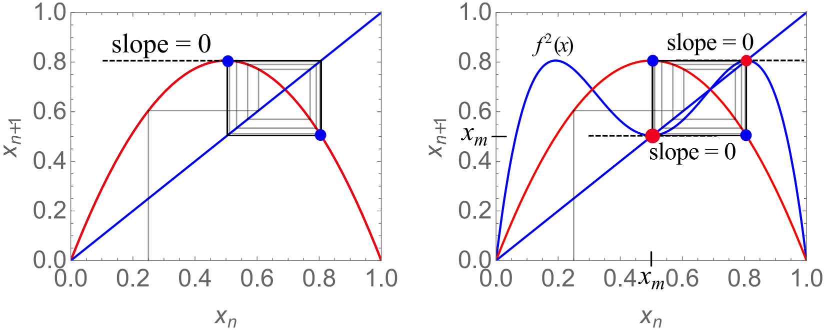
Convergence of x_n about the non-trivial fixed point x^*

$$\eta_{n+1} = \frac{|f''(x^*, r)|}{2!}\eta_n^2 + O(\eta_n^3), \qquad (4)$$

is quadratic. The iterates x_n converge quadratically.

Superstable period-p point (the logistic map)

Superstable period-p orbit: $x^* = x_m = \max f$ is also a local min. or max. of f^p map (p-th iterate of map f).

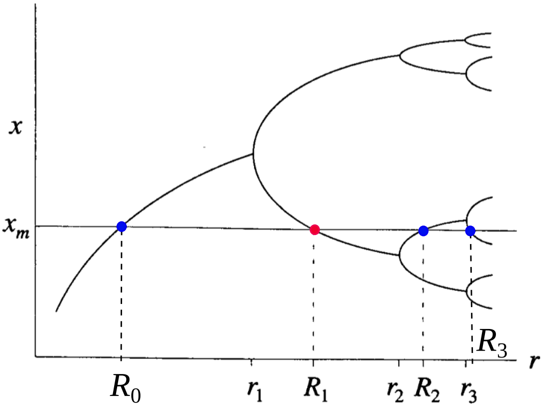


For example in the case of period-2 point

$$f^2(x^*, r) = x^*, \quad (x^* = x_m) \quad \Rightarrow \quad r = 1 + \sqrt{5} \tag{5}$$

$$\left(f^2(x^*, r)\right)' = \frac{\mathrm{d}}{\mathrm{d}x^*}[f(f(x^*, r))] = f'(f(x^*, r)) \cdot f'(x^*, r) = 0. \tag{6}$$

Orbit diagram, superstable period-2^n points



r_n – stable period-2^n orbit is born (bifurcation point).

R_n – superstable period-2^n point.

$$\lim_{n\to\infty} \frac{R_{n-1}}{R_n} = \delta \tag{7}$$

Period doubling bifurcation points

Values of bifurcation points and superstable points of a few first period doublings in the logistic map.

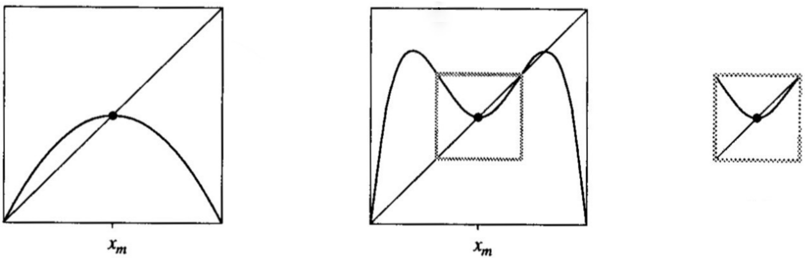
$r_0 = 1.0$ non-trivial	$R_0 = 2.0$	f.p.
$r_1 = 3.0$	$R_1 = 1 + \sqrt{5} \approx 3.23607$	period-2
$r_2 = 1 + \sqrt{6} \approx 3.44949$	$R_2 \approx 3.49856$	period-4
$r_3 \approx 3.54409$	$R_3 \approx 3.55464$	period-8
$r_4 \approx 3.56441$	$R_4 \approx 3.56667$	period-16
$r_5 \approx 3.56875$	$R_5 \approx 3.56924$	period-32
$\vdots$	$\vdots$	$\vdots$
$r_\infty \approx 3.569945672$	$R_\infty = r_\infty \approx 3.56994567$	period-2^∞

r_∞ – Onset of chaos (accumulation point).

$$\lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} = \lim_{n\to\infty} \frac{R_{n-1}}{R_n} = \delta \qquad (8)$$

Feigenbaum's analysis, renormalisation

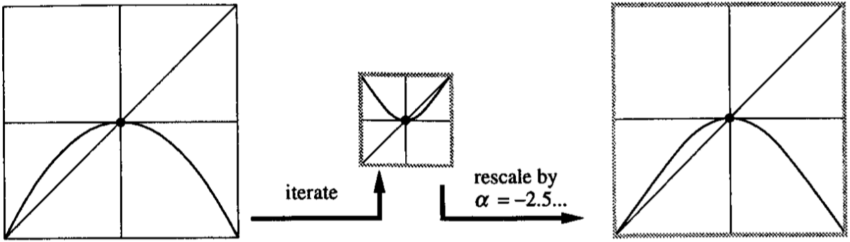
$f(x, R_0)$ $f^2(x, R_1)$



(a) (b) (c)

Feigenbaum's analysis, renormalisation

$f(x, R_0)$ $f^2(x, R_1)$ $\alpha f^2\left(\frac{x}{\alpha}, R_1\right)$



(a) (b) (c)

Read: Mitchell J. Feigenbaum, "The universal metric properties of nonlinear transformations," Journal of Statistical Physics 21(6), pp. 669–706, 1979. Also relevant to the following three slides ⇓.

Limiting function $g(x)$ and Feigenbaum constant α

Let's consider a functional equation in the following form:

$$g(x) = \alpha\, g(g\left(\frac{x}{\alpha}\right)) = \alpha\, g^2\left(\frac{x}{\alpha}\right), \tag{9}$$

where α acts as a scaling coefficient and

$$\alpha = \frac{1}{g(1)}. \tag{10}$$

The power series solution is obtained by assuming a power expansion in the following form:

$$g(x) = 1 + ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + \dots, \tag{11}$$

where the map maximum is quadratic.

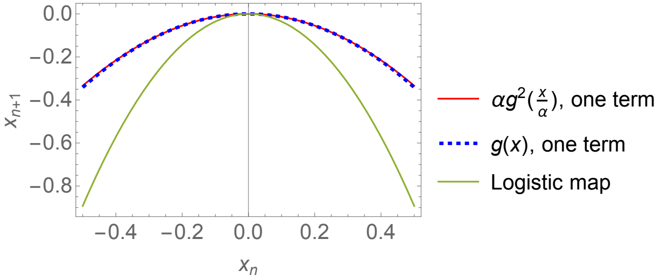
Limiting function $g(x)$ and Feigenbaum constant α

Numeric solution[1]: The one term approximation where

$$g(x) = 1 + ax^2 + O(x^4), \quad (12)$$

results in

$$a = -\frac{1}{2}(1+\sqrt{3}), \quad \alpha = \frac{1}{g(1)} \approx -2.73205, \quad (9.2\% \text{ error}). \quad (13)$$



[1]See Mathematica .nb file uploaded to course webpage.

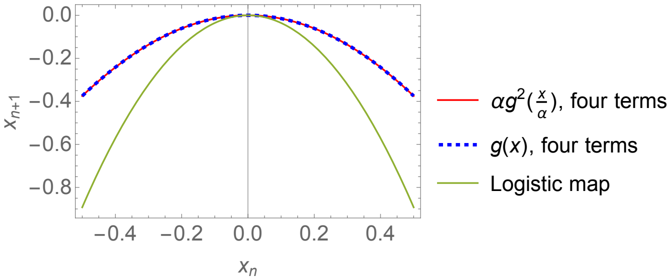
Limiting function $g(x)$ and Feigenbaum constant α

Numeric solution[2]**:** The four term approximation where

$$g(x) = 1 + ax^2 + bx^4 + cx^6 + dx^8 + O(x^{10}). \quad (14)$$

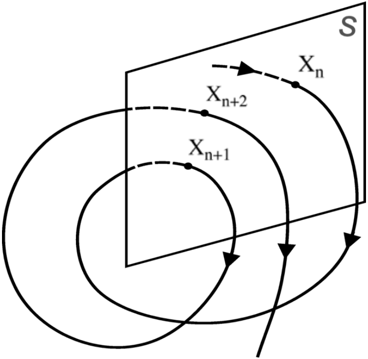
Coefficients: $a \approx -1.528$, $b \approx 0.1053$, $c \approx 0.02631$, $d \approx -0.003344$ and

$$\alpha = \frac{1}{g(1)} \approx -2.50316, \quad (0.01\% \text{ error}). \quad (15)$$



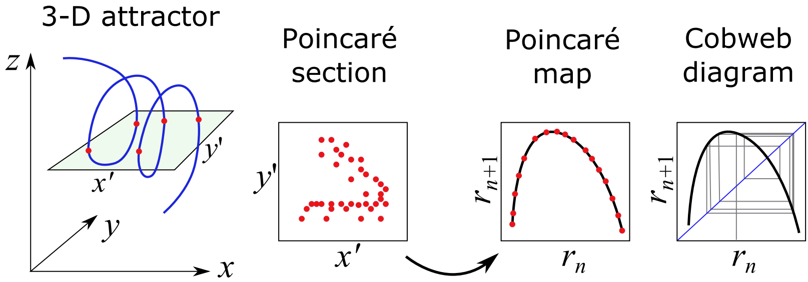
[2]See Mathematica .nb file uploaded to course webpage.

The Poincaré section



Intersection of an attractor trajectory with hypersurface S.

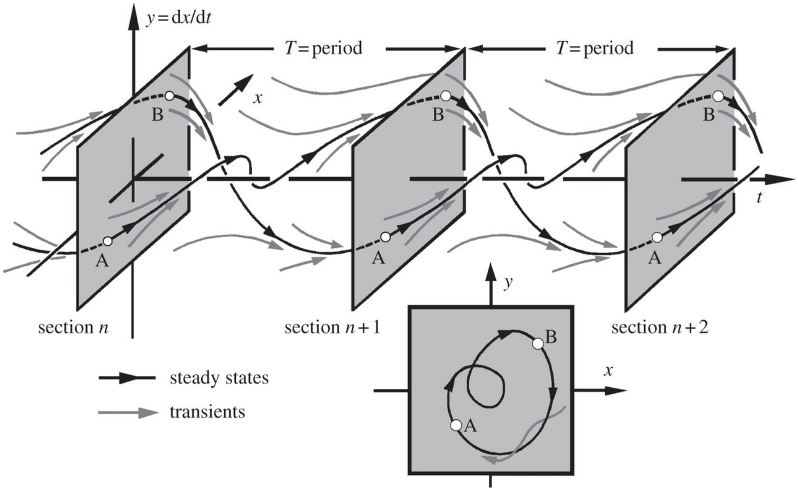
Discrete time dynamics analysis



$$\begin{cases} x'_{n+1} = f_1(x'_n, y'_n) \\ y'_{n+1} = f_2(x'_n, y'_n) \end{cases} \Rightarrow \quad r_{n+1} = f_3(r_n) \tag{16}$$

Construction of the Poincaré map $\vec{P}(\vec{x}') = (f_1(x', y'), f_2(x', y'))^T$ (16). Mapping of the Poincaré section points where r is the radial distance from the origin ("flat" attractor).

The Poincaré section, periodic forcing[3]



[3] Credit: J. Thompson, H. Stewart, 1986, *Nonlinear dynamics and chaos: geometrical methods for engineers and scientists*. Chichester, UK: Wiley.

The Poincaré section, periodic forcing

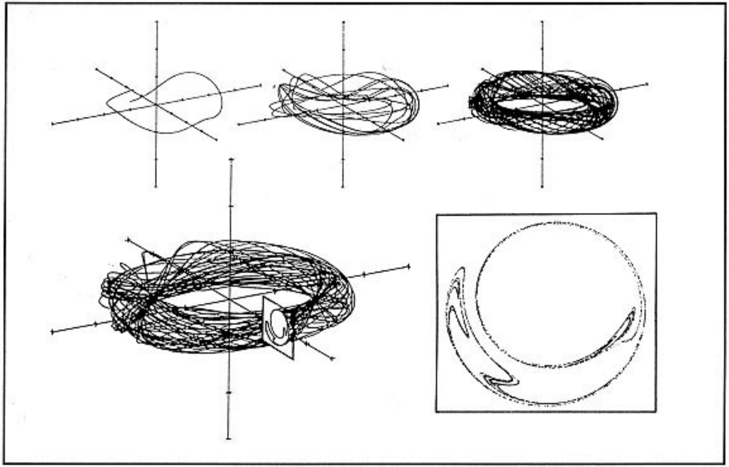


Figure 3. Illustration of a Strange Attractor and the Associated Poincaré Map (from James Gleick, *Chaos: Making a New Science* [New York: Pengin Books 1987], 143)

The periodically driven damped Duffing oscillator

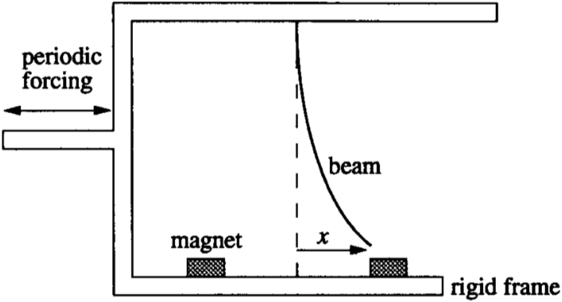


Figure: The mechanical Duffing oscillator.

The non-autonomous equation of motion has the following form:

$$\ddot{x} - x + x^3 + \delta\dot{x} = F\cos\omega t, \tag{17}$$

where δ is the damping coefficient, F and ω are the forcing strength and frequency, respectively.

The periodically driven damped Duffing oscillator

We introduce the variable exchange $y = \dot{x}$ and rewrite (17)

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3 - \delta y + F\cos\omega t. \end{cases} \tag{18}$$

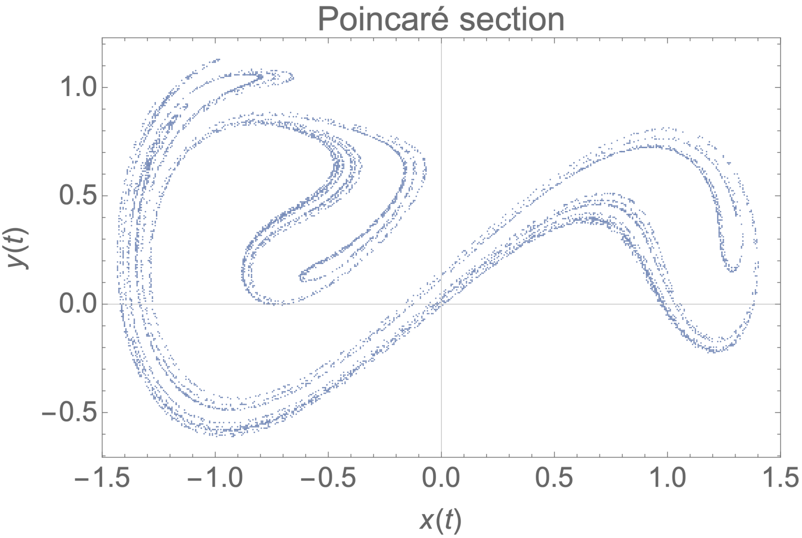
The equivalent 3-D system is obtained by applying variable exchange $z = \omega t$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3 - \delta y + F\cos z, \\ \dot{z} = \omega. \end{cases} \tag{19}$$

The above holds since

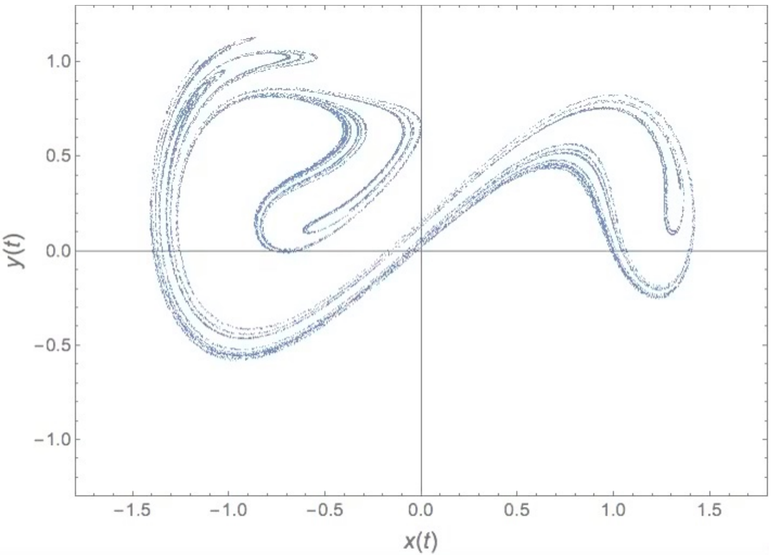
$$z = \int \dot{z}\,\mathrm{d}t = \int \omega\,\mathrm{d}t = \omega \int \mathrm{d}t = \omega t + C. \tag{20}$$

The Poincaré section[4] of the Duffing oscillator



[4] See Mathematica .nb file uploaded to the course webpage.

The Poincaré section dynamics: Duffing oscillator

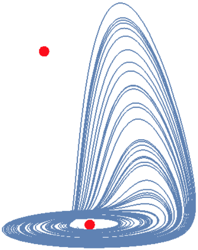


The Rössler attractor

As mentioned in Lecture 9 the Rössler attractor[5] is given by:

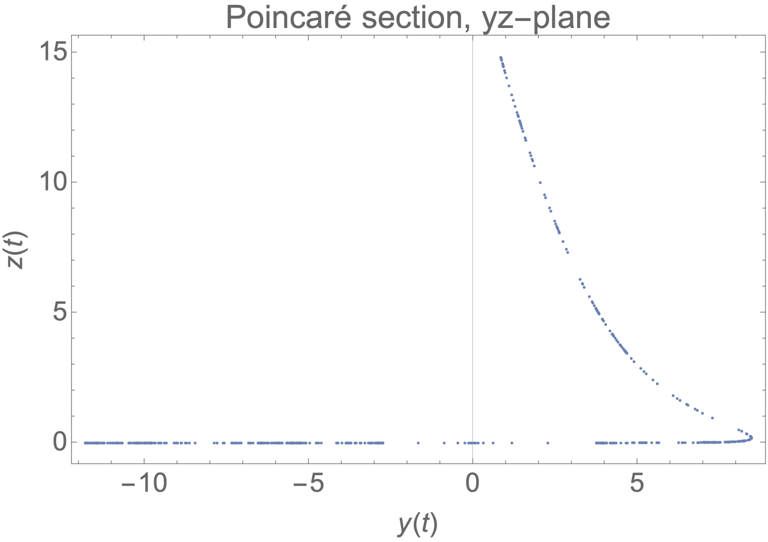
$$\begin{cases} \dot{x} = -y - x, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases} \tag{21}$$

Chaotic solution exists for $a = 0.1$, $b = 0.1$, $c = 14$.



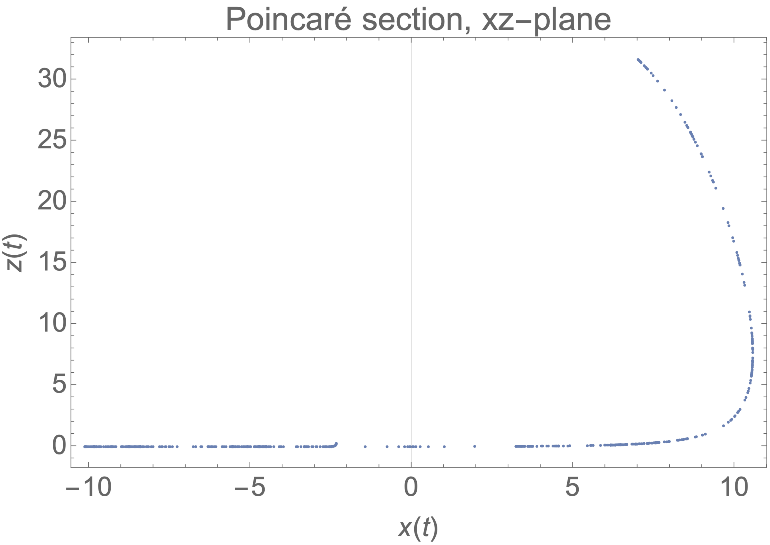
[5] See Mathematica .nb file uploaded to the course webpage.

The Rössler attractor: the Poincaré sections[6]



[6]See Mathematica .nb file uploaded to the course webpage.

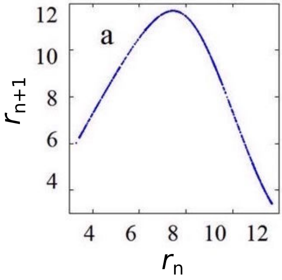
The Rössler attractor: the Poincaré sections[7]



[7]See Mathematica .nb file uploaded to the course webpage.

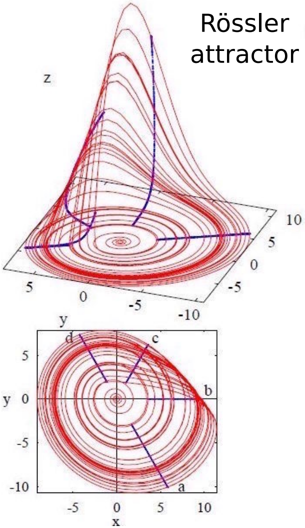
Rössler system: return map and Poincaré sections

Rössler map



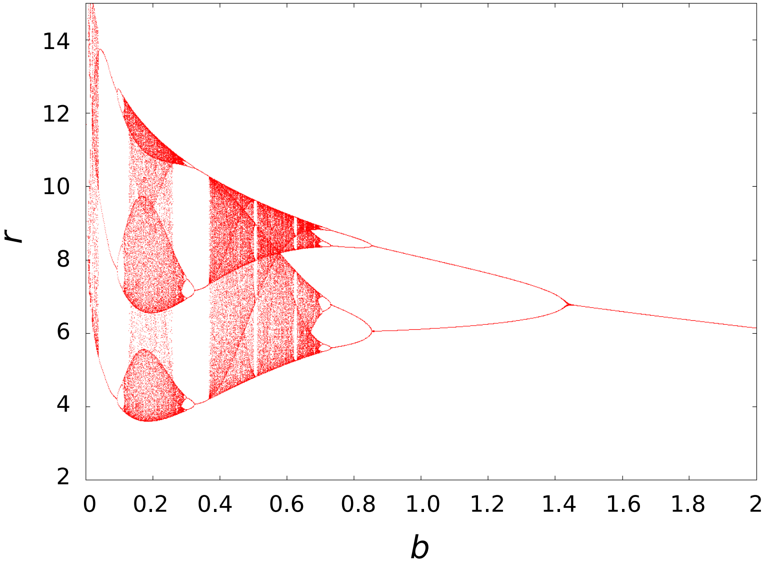
r is the radial distance from the origin

Rössler attractor

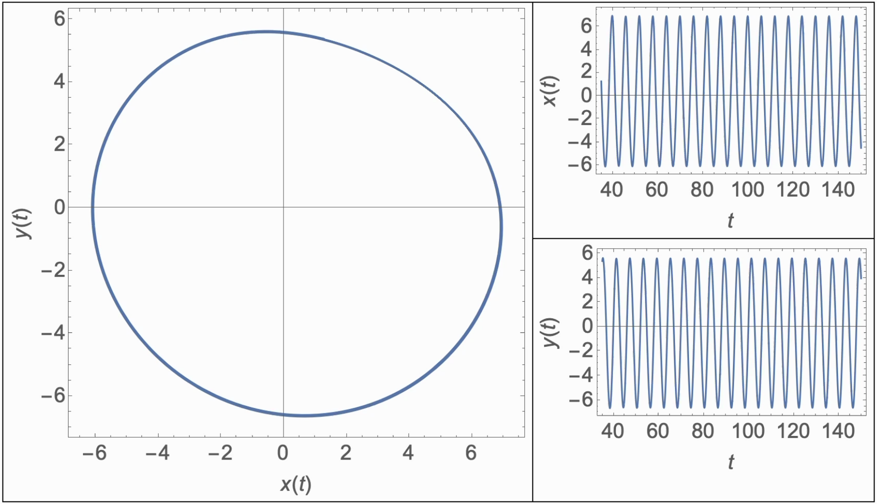


Credit: Y. Maistrenko and R. Paškauskas

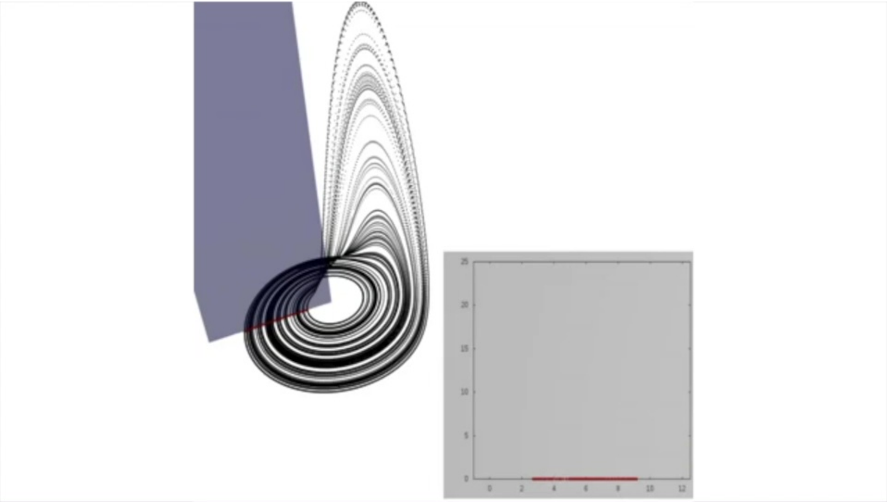
The Rössler attractor: an orbit diagram



The Rössler attractor: period doubling

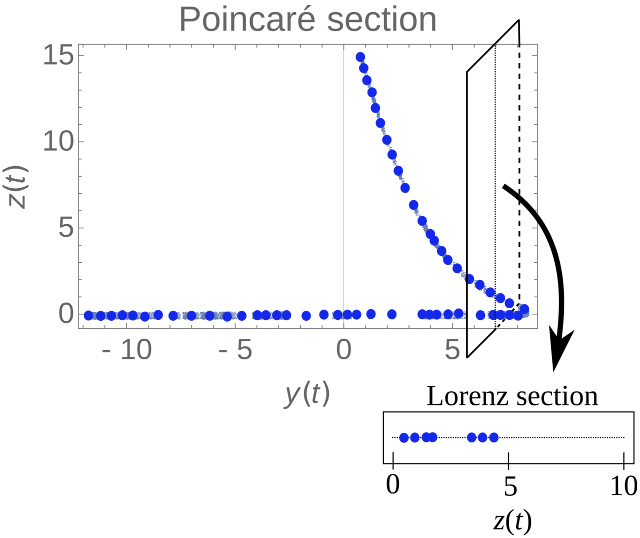


The Rössler attractor: the Poincaré section



Credit: Timothy Jones, `http://www.physics.drexel.edu/~tim/`

The Lorenz section



The Lorenz section — a section of a section.

Conclusions

- Feigenbaum's analysis of period doubling
- The universal route to chaos
- Universal aspects of period doubling in unimodal maps
- Superstable fixed points and period-p orbits
- Renormalisation
- Universal limiting functions and the onset of chaos
- Discrete-time dynamics analysis methods
    - The Poincaré section
    - Return map or the Poincaré map
    - The Lorenz section
- Non-homogenous systems
- Examples studied:
    - The periodically forced damped Duffing oscillator
    - The Rössler attractor

Revision questions

- What are the values of the Feigenbaum constants?
- What are the Feigenbaum constants?
- Define superstable fixed point of a map.
- Define superstable period-p point (or period-p orbit) of a map.
- What are universals of unimodal maps?
- What is the universal route to chaos?
- Idea behind renormalisation?
- What are the universal limiting functions in the context of maps?
- Name discrete-time dynamics analysis methods.
- What is the Poincaré section?
- What is the Poincaré map (return map)?
- What is the Lorenz section?