Lecture №10: 1-D unimodal maps, the Lorenz, the logistic and sine maps, period doubling bifurcation, tangent bifurcation, intermittency, orbit diagram, the Feigenbaum constants

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Lecture outline

- The Lorenz map and unstable limit-cycles, graphical approach
- Connection between 3-D chaotic systems and 1-D maps
- Period-p points (period-p orbit)
- The logistic map
- Analysis and properties of the logistic map
- Sine map
- Period doubling bifurcation in unimodal maps
- Tangent bifurcation in unimodal maps
- Orbit diagram (or the Feigenbaum diagram) or fig tree diagram
- The Feigenbaum diagram
- Universal aspect of period doubling in unimodal maps
- Universal route to chaos
- The Feigenbaum constants δ and α

The logistic map

The logistic map[1] has the following form:

$$x_{n+1} = rx_n(1 - x_n), \quad x_0 \in [0, 1], \quad r \in [0, 4], \quad n \in \mathbb{Z}^+, \qquad (1)$$

where r is the control parameter.

Read: Robert M. May, "Simple mathematical models with very complicated dynamics," *Nature* **261**, pp. 459–467, 1976. doi:10.1038/261459a0

[1]See Mathematica .nb file (cobweb diagram and orbit diagram) uploaded to the course webpage.

The Lyapunov exponent of the logistic map

Chaos is characterised by **sensitive dependence on initial conditions**. If we take two close-by initial conditions, say x_0 and $y_0 = x_0 + \eta$ with $\eta \ll 1$, and iterate them under the map, then the difference between the two time series $\eta_n = y_n - x_n$ should grow exponentially

$$|\eta_n| \sim |\eta_0 e^{\lambda n}|, \tag{2}$$

where λ is the Lyapunov exponent. For maps, this definition leads to a very simple way of measuring the Lyapunov exponents. Solving (2) for λ yields:

$$\lambda = \frac{1}{n} \ln \left| \frac{\eta_n}{\eta_0} \right|. \tag{3}$$

By definition $\eta_n = f^n(x_0 + \eta_0) - f^n(x_0)$. Thus

$$\lambda = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \eta_0) - f^n(x_0)}{\eta_0} \right|. \tag{4}$$

The Lyapunov exponent of the logistic map

For small values of η_0, the quantity inside the absolute value signs is just the derivative of f^n with respect to x evaluated at $x = x_0$:

$$\lambda = \frac{1}{n} \ln \left| \frac{\mathrm{d} f^n}{\mathrm{d} x} \right|_{x=x_0}. \tag{5}$$

Since, $f^n(x) = f(f(f(\ldots f(x)))\ldots)$, by the chain rule:

$$\begin{aligned} \left| \frac{\mathrm{d} f^n}{\mathrm{d} x} \right|_{x=x_0} &= \left| f'(f^{n-1}(x_0)) \cdot f'(f^{n-2}(x_0)) \cdot \ldots \cdot f'(x_0) \right| \\ &= |f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot \ldots \cdot f'(x_0)| = \left| \prod_{i=0}^{n-1} f'(x_i) \right|. \end{aligned} \tag{6}$$

Our expression for the Lyapunov exponent takes the form:

$$\lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{7}$$

The Lyapunov exponent of the logistic map

$$\lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$

The Lyapunov exponent is the large iterate n limit of this expression, and so we have:

$$\boxed{\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.} \tag{8}$$

This formula can be used to study the Lyapunov exponent[2] as a function of control parameter r:

$$\boxed{\lambda(r) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i, r)|.} \tag{9}$$

[2] See Mathematica .nb file uploaded to the course webpage.

The logistic map, period-2 window

Period-2 window for $3 \leq r < 1 + \sqrt{6}$.

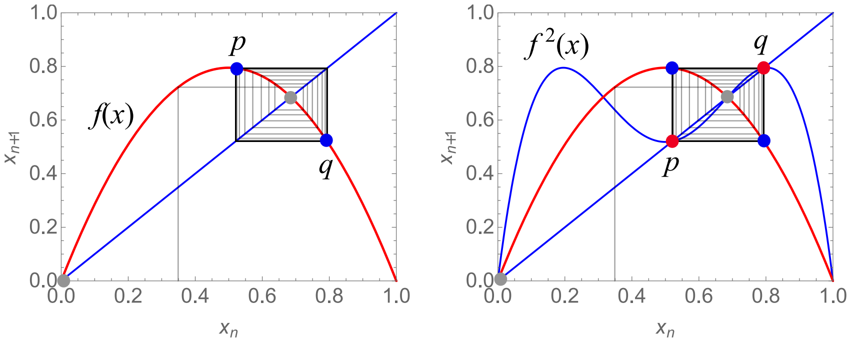


Figure: The logistic map where $r = 3.18$ and $x_0 = 0.35$. Fixed points (f.p.s) in the case where $r < 3$ are shown with the grey bullets. Period-2 points of $f(x)$ map for $r \geq 3$ are shown with the blue bullets. The fixed points of $f^2(x)$ map for $r \geq 3$ are shown with the red bullets.

The logistic map, period-2 window

Period-2 window for $3 \leq r < 1 + \sqrt{6}$.

$$\begin{cases} f(p) = rp(1-p) = q, \\ f(q) = rq(1-q) = p, \end{cases} \tag{10}$$

here period-2 point values p and q are the f.p.s of $f(x)$ map.

On the other hand it also holds

$$\begin{cases} f(p) = f(f(q)) \equiv f^2(q) = r[rq(1-q)][1-(rq(1-q))] = q, \\ f(q) = f(f(p)) \equiv f^2(p) = r[rp(1-p)][1-(rp(1-p))] = p, \end{cases} \Rightarrow \tag{11}$$

$$\Rightarrow \quad f^2(x) = r[rx(1-x)][1-(rx(1-x))] = x, \tag{12}$$

where period-2 point values p and q are the f.p.s of $f^2(x)$ map.

Stability of f.p.s of f^2 map in period-2 orbit

We need to know the slopes of period-2 points

$$\begin{cases} f(p) = rp(1-p) = q, \\ f(q) = rq(1-q) = p. \end{cases}$$

According to the chain rule it holds that

$$(f^2(x))' \equiv (f(f(x))' = f'(f(x)) \cdot f'(x). \tag{13}$$

In our case:

$$\left.\begin{matrix} (f^2(p))' = f'(f(p)) \cdot f'(p) = f'(q) \cdot f'(p) \\ (f^2(q))' = f'(f(q)) \cdot f'(q) = f'(p) \cdot f'(q) \end{matrix}\right\} \Rightarrow (f^2(p))' = (f^2(q))'. \tag{14}$$

Above follows from the commutative property of multiplication.

The logistic map, period doubling

Even number periods.

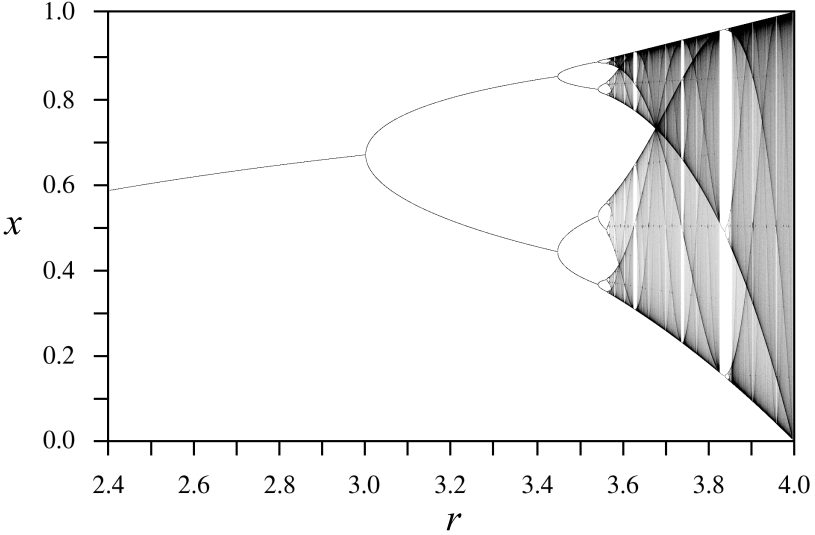
r_n – bifurcation point, onset of a stable period-2^n orbit.

$r_1 = 3.0$	period-2
$r_2 = 1 + \sqrt{6} \approx 3.44949$	period-4
$r_3 \approx 3.54409$	period-8
$r_4 \approx 3.56441$	period-16
$r_5 \approx 3.56875$	period-32
$r_6 \approx 3.56969$	period-64
$\vdots$	$\vdots$
$r_\infty \approx 3.569946$	period-2^∞

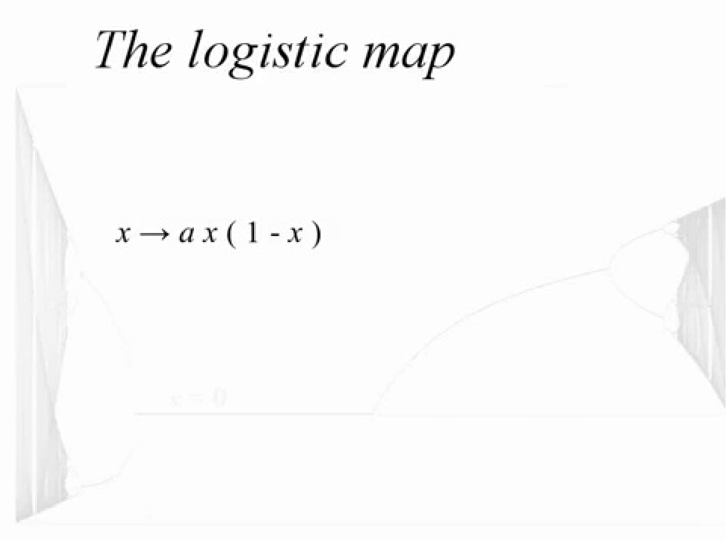
r_∞ – onset of chaos (the accumulation point).

$$\delta = \lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609... \quad (15)$$

Orbit diagram and period doubling

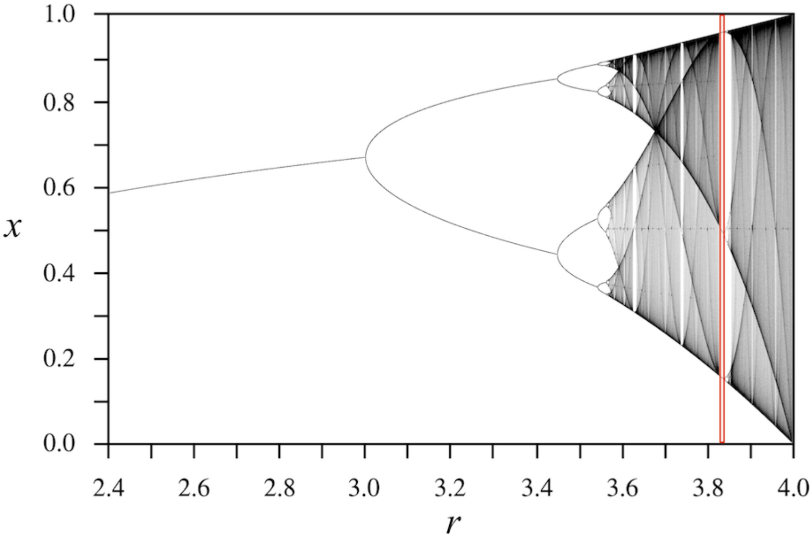


Zooming into the logistic map, self-similarity



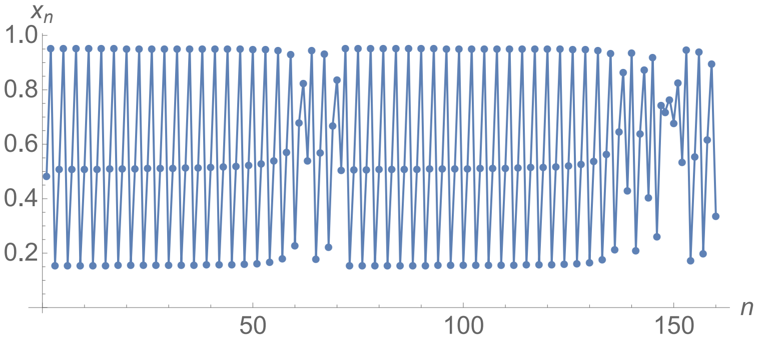
Credit: `https://www.youtube.com/user/logicedges/`

Orbit diagram, period-3 window



Intermittency[3] and period-3 window

Transient chaos and intermittency in dynamical systems. Tangent bifurcation occurs at $r = 1 + \sqrt{8} \approx 3.8284$ (onset of period-3 orbit).



Iterates of the logistic map shown for $r = 3.8282$ and $x_0 = 0.15$.

[3] See Mathematica .nb file uploaded to the course webpage.

Universality of period doubling in unimodal maps

1-D sine map[4]. The sine map has the form:

$$x_{n+1} = r\sin(\pi x_n), \quad x_0 \in [0,1], \quad r \in [0,1], \quad n \in \mathbb{Z}^+, \tag{16}$$

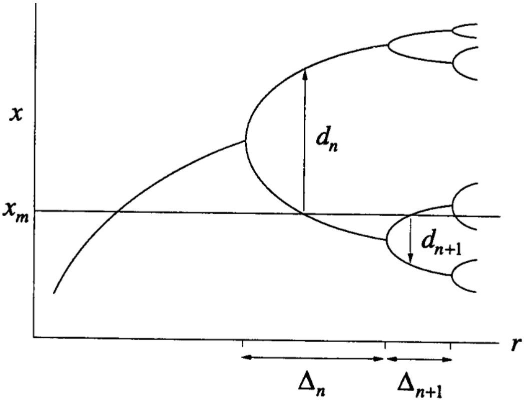
where r is the control parameter.

Read: Mitchell J. Feigenbaum, "Quantitative universality for a class of nonlinear transformations," *Journal of Statistical Physics* **19**(1), pp. 25–52, 1978, `doi:10.1007/BF01020332`

Read: Mitchell J. Feigenbaum, "Universal behavior in nonlinear systems," *Physica D: Nonlinear Phenomena* **7**(1–3), pp. 16–39, 1983, `doi:10.1016/0167-2789(83)90112-4`

[4] See Mathematica .nb file (cobweb diagram and orbit diagram) uploaded to the course webpage.

1-D unimodal maps and the Feigenbaum constants



$$\delta = \lim_{n\to\infty} \frac{\Delta_{n-1}}{\Delta_n} = \lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609... \qquad (17)$$

$$\alpha = \lim_{n\to\infty} \frac{d_{n-1}}{d_n} \approx -2.502907875... \qquad (18)$$

Conclusions

- The Lorenz map and unstable limit-cycles, graphical approach
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Revision questions

- What is cobweb diagram?
- What is recurrence map or recurrence relation?
- What is 1-D map?
- How to find fixed points of 1-D maps?
- What is the Lorenz map?
- What is the logistic map?
- What is sine map?
- What is period doubling?
- What is period doubling bifurcation?
- What is tangent bifurcation?
- Do odd number periods (period-p orbits) exist in chaotic systems?
- Do even number periods (period-p orbits) exist in chaotic systems?

Revision questions

- Can maps produce transient chaos?
- Can maps produce intermittency?
- Can maps produce intermittent chaos?
- What is orbit diagram (or the Feigenbaum diagram)?
- What are the Feigenbaum constants?