Lecture 7: Classification of bifurcations in 2-D systems, bifurcations of fixed points, the Hopf bifurcation, the supercritical and subcritical Hopf bifurcations, bifurcations of closed orbits, hysteresis on the level of cycles, examples of dynamical instabilities

Contents

Handout: Classification of bifurcations in 2-D systems.

1 Bifurcations in 2-D systems

This lecture extends our earlier work on bifurcations, see Lecture 2. As we move up from one-dimensional to two-dimensional systems, we still find that fixed points can be created or destroyed or destabilised as parameters are varied—but now the same is true of closed orbits and limit-cycles as well. Thus we can begin to describe the ways in which oscillations can be turned on or off.

In this broader context, what exactly do we mean by a bifurcation? The *usual* definition involves the concept of **topological equivalence**: if the phase portrait changes its topological structure (loses topological equivalence) as a parameter is varied, we say that a bifurcation has occurred. Examples include changes in the number or stability of fixed points, closed orbits, or saddle connections as a parameter is varied. **Intuitively**, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

2 Classification of bifurcations in 2-D systems

Here we present a selection of bifurcations that most commonly occur in practical applications. Bifurcations in two-dimensional systems can be classified for example as follows:

CASE I Bifurcations of fixed points

- A) Bifurcations at $\lambda_1 = 0$ or $\lambda_2 = 0$ where $\lambda_{1,2}$ are the system eigenvalues.
  - 1) Saddle-node bifurcation
  - 2) Transcritical bifurcation
  - 3) Pitchfork bifurcation
    - ∗ Supercritical pitchfork bifurcation
    - ∗ Subcritical pitchfork bifurcation
- B) The Hopf bifurcations, bifurcations at $\lambda_{1,2} = \pm i\omega$
  - 1) The supercritical Hopf bifurcation
  - 2) The subcritical Hopf bifurcation

CASE II Global bifurcations of closed orbits

- A) Saddle-node coalescence of cycles (accompanied by the subcritical Hopf)
- B) SNIPER (saddle-node infinite period bifurcation) also called SNIC (saddle-node in invariant cycle bifurcation)
- C) Homoclinic bifurcation or saddle-loop bifurcation

This lecture is organised as follows: for each bifurcation, we present and discuss the normal form or a simple prototypical example describing the bifurcation. Interactive numerical files are used to show the phase plane dynamics as the bifurcation parameter is varied.

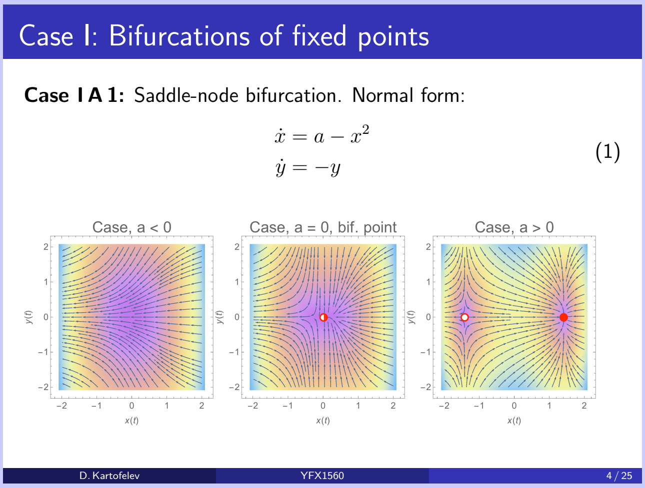
3 CASE I A 1: Saddle-node bifurcation

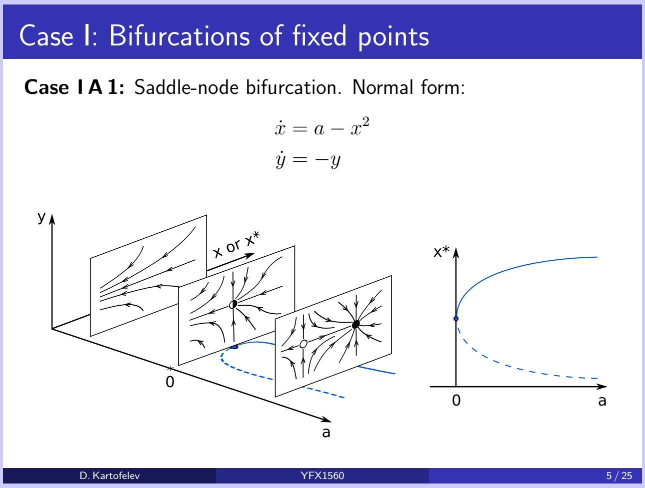
The bifurcations of fixed points discussed in Lecture 2 have analogs in two dimensions, and indeed, in all dimensions. Yet it turns out that nothing really new happens when more dimensions are added—all the action is confined to a one-dimensional subspace along which the bifurcations occur, while in the extra dimensions the flow is either simple attraction to or repulsion from that subspace, as we see below on Slide 5.

Suppose a two-dimensional system has a stable fixed point. What are all the possible ways it could lose stability as a parameter is varied? The eigenvalues λ_i of the Jacobian evaluated at that fixed point or in other words the eigenvalues λ_i of **linearised system**, are the key (see Lecture 4). Since, the λ's satisfy a quadratic characteristic equation with real coefficients, there are two possible pictures: either the eigenvalues are **both real and negative** or they are **complex conjugates**, see the classification chart for fixed points in 2-D linear systems. If the fixed point is stable, the eigenvalues λ_1 and λ_2 must both $\lambda_1, \lambda_2 < 0$ or $\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2 < 0$. To destabilise the fixed point, we need one or both of the eigenvalues to become **positive** as the bifurcation parameter changes.

SLIDES: 4, 5

The **saddle-node bifurcation** is the basic mechanism for creation and destruction of fixed points.





On x-axis we see the bifurcation behaviour discussed in Lecture 2 (1-D dynamics), while in y-direction the motion is exponentially damped and approaching the x-axis.

Case $a < 0$ shown on Slide 4 features a "**ghost**"—a bottleneck region of the phase portrait where the flow on trajectories is slowed down. The slower vector field velocities are colour-coded with the purple colour. The *ghost* is predicting the bifurcation—the appearance of the half-stable fixed point $(x^*, y^*) = (0, 0)$ for $a = 0$.

Figure 1 shows fixed point eigenvalues in the $\mathrm{Im}\,\lambda$-$\mathrm{Re}\,\lambda$ graph corresponding to a saddle-node bifurcation. Once again, if one or both λ's cross into the right half-plane of the graph the stability of fixed point is changed from stable to unstable, *cf.* Figs 1, 2, 3 and 4. Point $\lambda_i = 0$ corresponds to the bifurcation point. The saddle-node, transcritical and pitchfork bifurcations occur at $\lambda_1 = 0$ or $\lambda_2 = 0$.

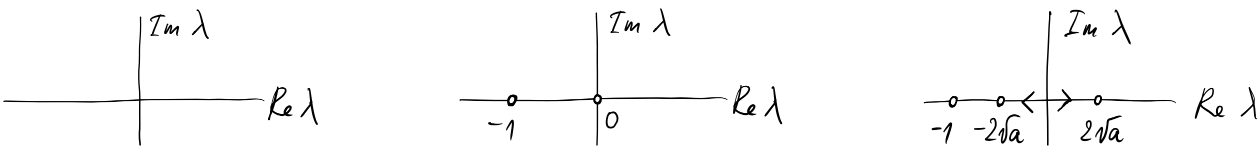


Figure 1: Eigenvalues of the Jacobian evaluated at fixed the points that participate in saddle-node bifurcation. (Left) Case where the bifurcation parameter $a < 0$ with no real valued fixed points, see Slides 4, 5. (Middle) Case $a = 0$, the bifurcation point. (Right) Case $a > 0$, featuring a saddle and a node.

The following interactive numerical file shows the phase plane dynamics of the saddle-node bifurcation.

NUMERICS: NB#1

Examples of bifurcations in 2-D systems: saddle-node bifurcation, transcritical bifurcation, supercritical and subcritical pitchfork bifurcations. Interactive code.

The full linear analysis of fixed points undergoing a saddle-node bifurcation is presented here.

NUMERICS: NB#2

Linear analysis of the system from the previous .nb file.

4 CASE I A 2: Transcritical bifurcation

Slide 6 shows transcritical bifurcation occurring. Here again, we see that in the x-direction the bifurcation behaviour is identical to the corresponding 1-D system dynamics presented in Lecture 2, and in the y-direction the motion is exponentially damped and approaching the x-axis. Same conclusion will hold for the **pitchfork bifurcations** discussed in the next section.

SLIDE: 6

In a **transcritical bifurcation** the fixed points are not destroyed whereas their stability is changed.

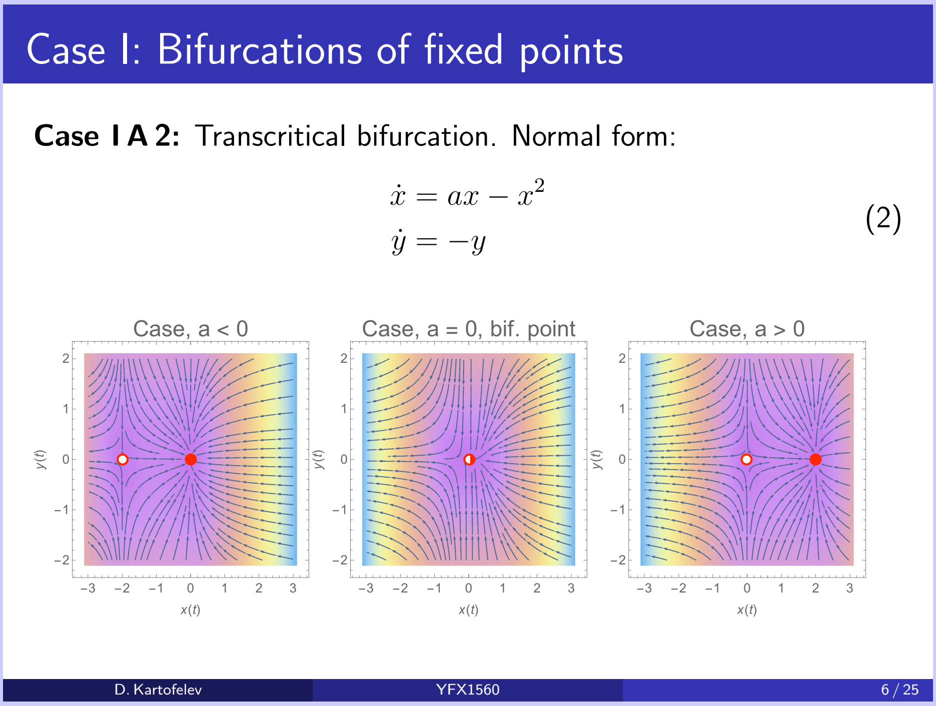


Figure 2 shows the eigenvalues of the system Jacobian evaluated at the fixed points undergoing a transcritical bifurcation.

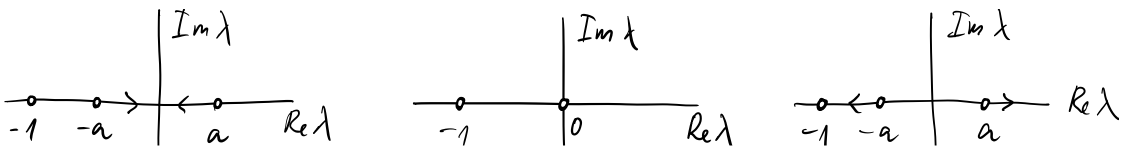


Figure 2: Eigenvalues of the Jacobian evaluated at the fixed points that are participating in a transcritical bifurcation. (Left) Case where the bifurcation parameter $a < 0$. (Middle) Case $a = 0$, the bifurcation point. (Right) Case $a > 0$.

The following interactive numerical file shows the phase plane dynamics of a transcritical bifurcation.

NUMERICS: NB#1

Examples of bifurcations in 2-D systems: saddle-node bifurcation, transcritical bifurcation, supercritical and subcritical pitchfork bifurcations. Interactive code.

Linear analysis of fixed points undergoing a transcritical bifurcation is presented in the following numerical file. Also, the file features the derivation of the eigenvalues shown in Fig. 2.

NUMERICS: NB#2

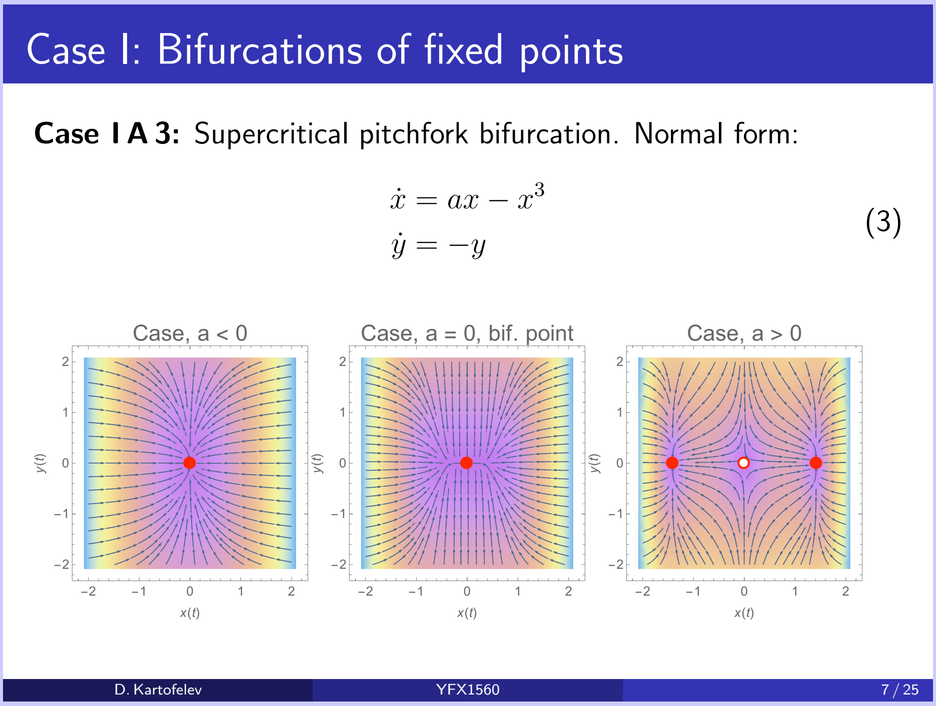
Linear analysis of the system from the previous .nb file.

5 CASE I A 3: Pitchfork bifurcation

5.1 Supercritical pitchfork bifurcation

SLIDE: 7

Pitchfork bifurcations occur in systems with symmetry. In the case of supercritical pitchfork bifurcation the bifurcating pitchfork branches are stable fixed points.



Case $a = 0$ features **algebraic decay** or **critical slowing** of the flow as it approaches the fixed point.

Figure 3 shows the eigenvalues of the system Jacobian evaluated at the fixed points undergoing a supercritical pitchfork bifurcation.

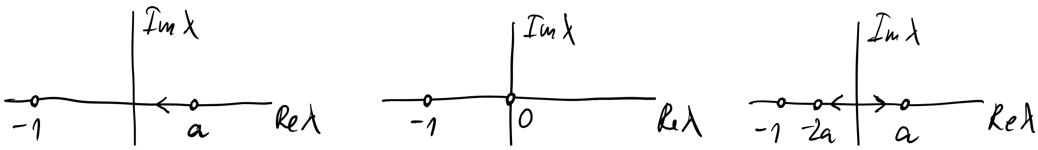


Figure 3: Eigenvalues of the Jacobian evaluated at the fixed points that participate in a supercritical pitchfork bifurcation. (Left) Case where the bifurcation parameter $a < 0$. (Middle) Case $a = 0$, the bifurcation point. (Right) Case $a > 0$, featuring an unstable saddle at the origin and two bifurcated nodes.

The following interactive numerical file shows the phase plane dynamics of a supercritical pitchfork bifurcation.

NUMERICS: NB#1

Examples of bifurcations in 2-D systems: saddle-node bifurcation, transcritical bifurcation, supercritical and subcritical pitchfork bifurcations. Interactive code.

Linear analysis of fixed points undergoing a supercritical pitchfork bifurcation is presented in the following numerical file. Also, the file features the derivation of the eigenvalues shown in Fig. 3.

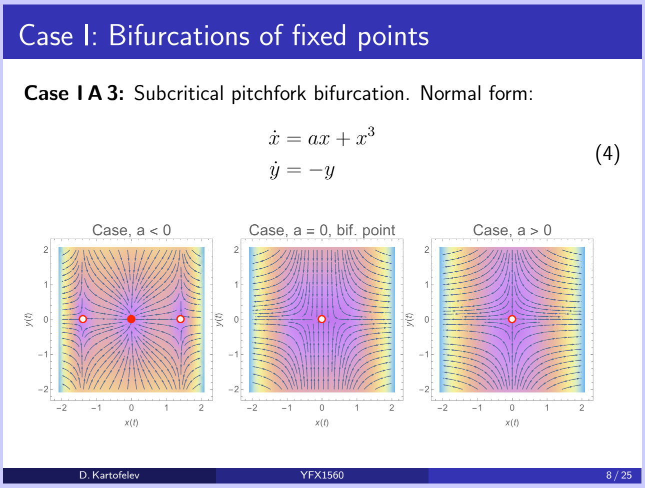
NUMERICS: NB#2

Linear analysis of the system from the previous .nb file.

5.2 Subcritical pitchfork bifurcation

SLIDE: 8

In the case of subcritical pitchfork bifurcation the bifurcating branches are <u>unstable</u> fixed points.



Case $a = 0$ features **algebraic decay** or **critical slowing** of the flow near the fixed point.

Figure 4 shows the eigenvalues of the system Jacobian evaluated at the fixed points undergoing a subcritical pitchfork bifurcation.

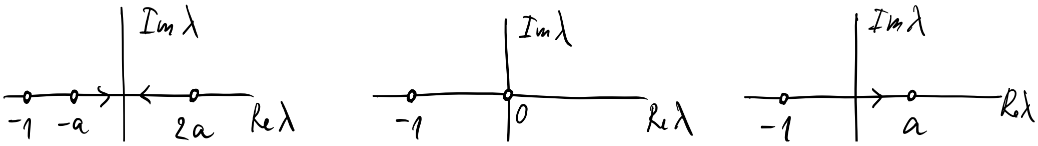


Figure 4: Eigenvalues of the Jacobian evaluated at the fixed points that are participating a subcritical pitchfork bifurcation. (Left) Case where the bifurcation parameter $a < 0$, featuring two unstable bifurcated saddles with a stable node at the origin. (Middle) Case $a = 0$, the bifurcation point. (Right) Case $a > 0$.

The following interactive .nb file shows the phase plane dynamics of a subcritical pitchfork bifurcation.

NUMERICS: NB#1

Examples of bifurcations in 2-D systems: saddle-node bifurcation, transcritical bifurcation, supercritical and subcritical pitchfork bifurcations. Interactive code.

Linear analysis of fixed points undergoing a subcritical pitchfork bifurcation is presented in the following numerical file. Also, the file features the derivation of the eigenvalues shown in Fig. 4.

NUMERICS: NB#2

Linear analysis of the system from the previous .nb file.

6 CASE I B 1: The supercritical Hopf bifurcation

In CASE I A we explored the cases in which a **real eigenvalue** passes through $\lambda_i = 0$ or Im λ-axis used in Figs 1, 2, 3 and 4. These were our old friends first presented during Lecture 2, namely the saddle-node, transcritical, and both types of pitchfork bifurcations.

Let's consider the other possible scenario, mentioned in Sec. 3, in which **two complex conjugate eigenvalues** simultaneously cross the imaginary axis in the Im λ-Re λ graph used above. The **normal form** of the supercritical Hopf bifurcation, given in <u>polar coordinates</u>, is the following:

$$\begin{cases} \dot{r} = \mu r - r^3, \\ \dot{\theta} = \omega + br^2, \end{cases} \tag{1}$$

where μ is the bifurcation parameter, b is the control parameter regulating amplitude dependency of the rotation rate $\dot{\theta}$, and ω is the angular velocity. If we assume $b \ll 1$, i.e., oscillation frequency $\dot{\theta}$ is weakly dependent on amplitude r, then the original system can be approximated by the following one:

$$\begin{aligned} \dot{r} &= \mu r - r^3, \\ \dot{\theta} &\approx \omega. \end{aligned} \tag{2}$$

This decoupled system is much more intuitive compared to Sys. (1). Let's study the dynamics of Sys. (2) in radial direction r and deduce the phase portrait for constant rotation rate $\dot{\theta} = \omega$. Figure 5 shows the dynamics for the stated assumptions and varied μ.

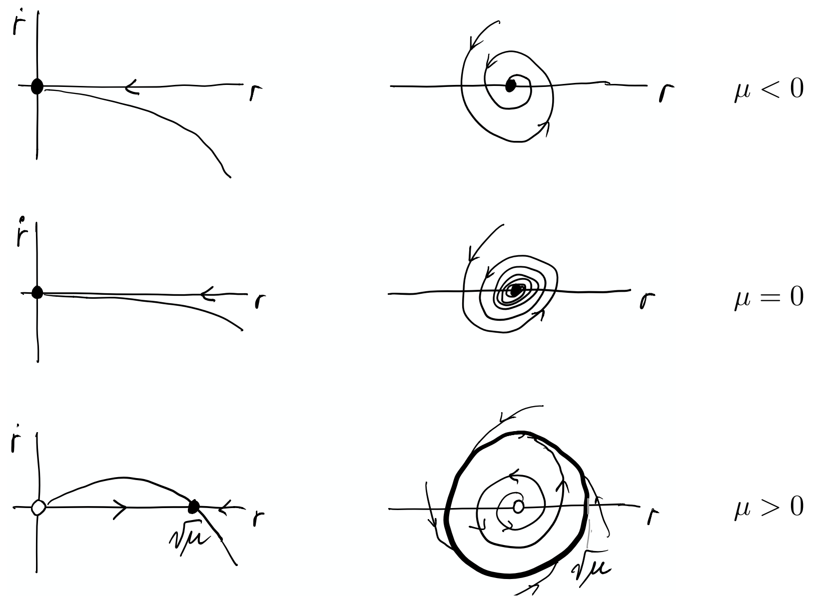
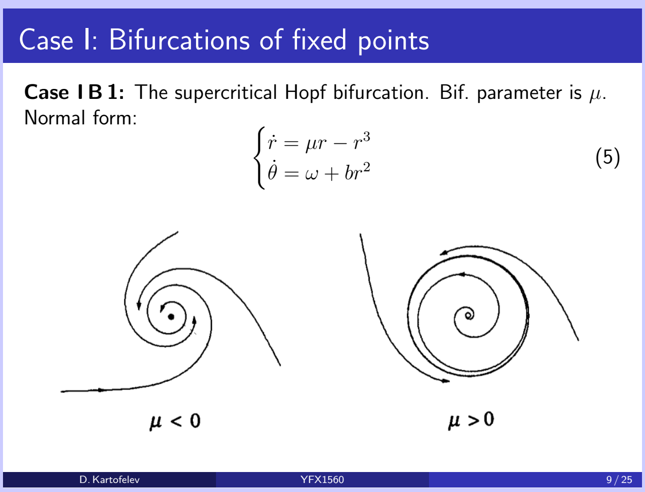


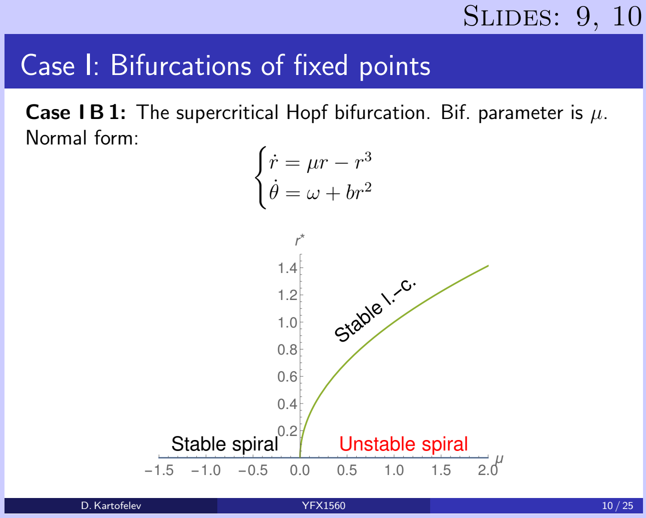
Figure 5: Dynamics in direction r and phase portraits for varied μ. (Top) Case $\mu < 0$. (Middle) Case $\mu = 0$, the bifurcation point. Trajectories approach the fixed point **algebraically**, the flow near fixed point is critically damped. (Bottom) Case $\mu > 0$, featuring a stable limit-cycle that is shown with the bold circle.

The above results can be checked against numerical calculations. The dynamics in radial direction, time-domain solutions, and phase portraits are shown in the following interactive numerical file.

> NUMERICS: NB#3
> Examples of bifurcations in 2-D systems: the supercritical Hopf bifurcation. Integrated solution and bifurcation diagram.

SLIDES: 9, 10





Slide 10 shows the bifurcation diagram directly corresponding to the results shown in Fig. 5. The supercritical bifurcation phenomena are always associated with **stable bifurcated objects**, in this case a stable limit-cycle. The stable limit-cycle is associated with the sign of r^3 term in Sys. (1).

Figure 6 shows the eigenvalues of the system Jacobian evaluated at the fixed point the origin undergoing a supercritical Hopf bifurcation.
Note: Don't try to obtain the eigenvalues in the polar coordinated. Convert Sys. (1) into the Cartesian coordinates and proceed from there.

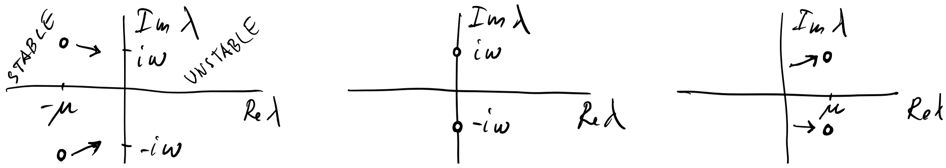


Figure 6: Eigenvalues of the Jacobian evaluated at the fixed point (located at the origin) that is participating in a supercritical Hopf bifurcation. (Left) Case where the bifurcation parameter $\mu < 0$. (Middle) Case $\mu = 0$, the bifurcation point. (Right) Case $\mu > 0$.

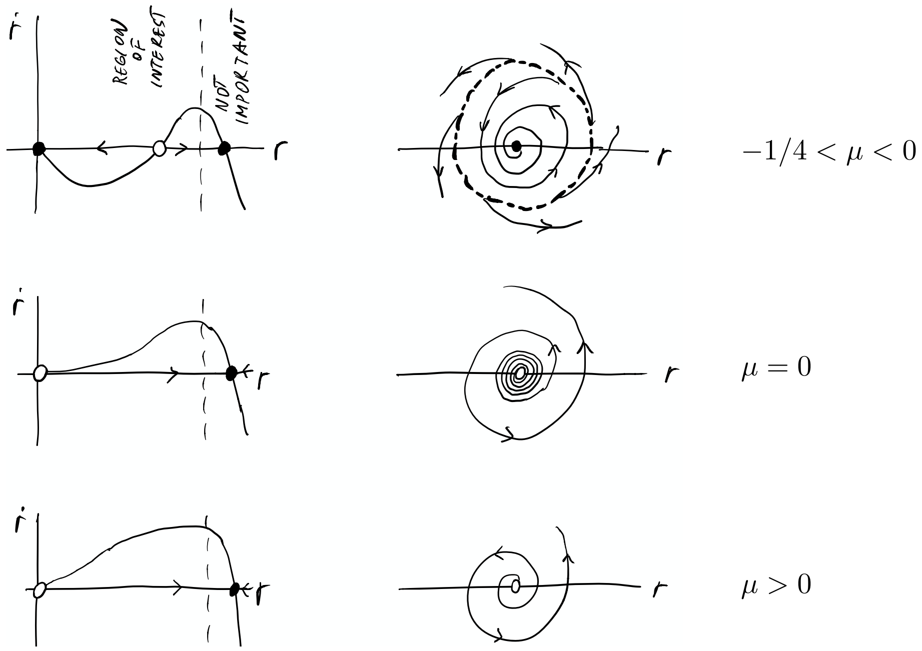


Figure 7: Dynamics in direction r and phase portraits for varied μ. (Top) Case $-1/4 < \mu < 0$. Dynamics for $\mu \leq -1/4$ is not part of this bifurcation. $\mu \leq -1/4$ corresponds to the dynamics shown to the right of the vertical dashed line in $\dot{r}(r)$ graphs. (Middle) Case $\mu = 0$, the bifurcation point. Trajectories are moving away from the fixed point **algebraically** where the flow is critically showed down. (Bottom) Case $\mu > 0$.

7 CASE I B 2: The subcritical Hopf bifurcation

Normal form of the subcritical Hopf bifurcation given in polar coordinates is the following:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5, \\ \dot{\theta} = \omega + b r^2, \end{cases} \tag{3}$$

where μ is the bifurcation parameter, b is the control parameter, carrying the same meaning as above, and ω is the angular velocity. Here again, we assume $b \ll 1$ and rewrite the original system as follows:

$$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5, \\ \dot{\theta} &\approx \omega. \end{aligned} \tag{4}$$

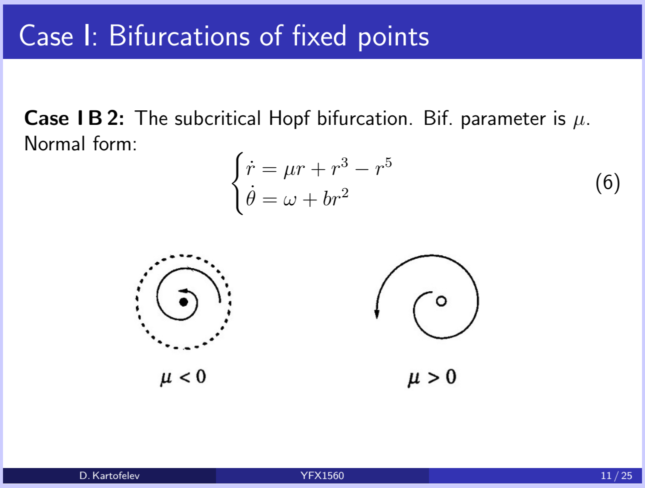
Let's study the dynamics of this system in radial direction r and deduce the phase portrait for constant rotation rate $\dot{\theta} = \omega$. Figure 7 shows the dynamics for the stated assumptions and varied μ. The dynamics shown to the right-hand side from the vertical dashed line in $\dot{r}(r)$ graphs in Fig. 7 is ignored here. This dynamics is part of the saddle-node coalescence of cycles bifurcation discussed in the next section and thus not a part of the subcritical Hopf bifurcation discussed here.

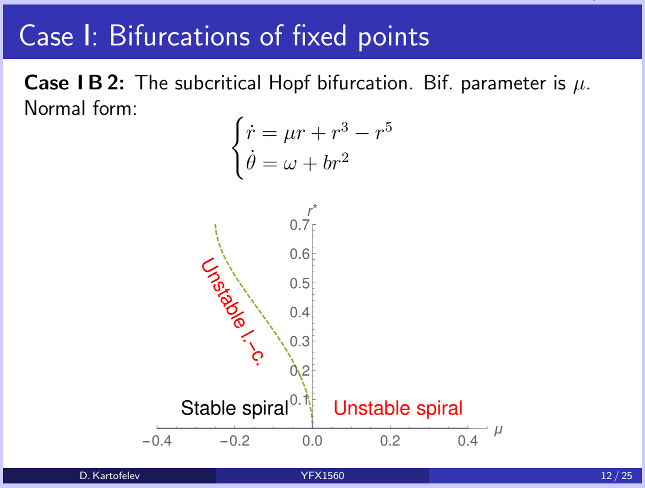
The above results can be checked against numerical calculations. The dynamics in radial direction, time-domain solutions, and the phase portraits are shown in the following interactive numerical file.

NUMERICS: NB#4

Examples of bifurcations in 2-D systems: the subcritical Hopf bifurcation. Integrated solution and bifurcation diagram.

SLIDES: 11, 12





Slide 12 shows the bifurcation diagram for r-direction directly corresponding to the results shown in Fig. 7. The subcritical bifurcation phenomena are alway associated with **unstable bifurcated objects**, in this case an unstable limit-cycle. The unstable limit-cycle is associated with the sign of r^3 term in Sys. (3).

Figure 8 shows the eigenvalues of the system Jacobian evaluated at the fixed point the origin undergoing a supercritical Hopf bifurcation. The behaviour of the eigenvalues is the same when compared to the supercritical case, shown in Fig. 6 and will be the same for CASE II A.

Note: Linear analysis via linearisation can't distinguish the supercritical Hopf from subcritical one.

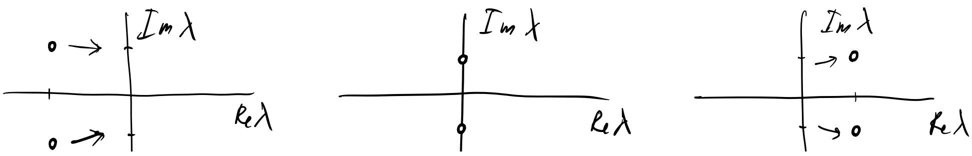


Figure 8: Eigenvalues of the Jacobian evaluated at the fixed point (located at the origin) that is participating in a subcritical Hopf bifurcation. (Left) Case where the bifurcation parameter $-1/4 < \mu < 0$. (Middle) Case $\mu = 0$, the bifurcation point. (Right) Case $\mu > 0$.

8 CASE II A: Saddle-node coalescence of cycles (+ subcritical Hopf)

An **example system** for a saddle-node coalescence of cycles bifurcation is the same as used above in the case of the subcritical Hopf bifurcation (3). We are reproducing it here again for clarity:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5, \\ \dot{\theta} = \omega + br^2, \end{cases}$$

here μ was the bifurcation parameter. Here too, we study the simplified approximated Sys. (4). Figure 9 shows the dynamics for varied μ. Bifurcation point $\mu = \mu_c = -1/4$ corresponds to the **coalescence of stable and unstable limit-cycles**. At this point a half-stable limit cycle is born or destroyed. The half-stable limit-cycle is stable from the outside and unstable from the inside. Trajectories approaching from outside will settle onto this limit-cycle but even the smallest perturbation will send any trajectory towards the stable spiral located at the origin. Notice that the origin remains stable during this bifurcation and also for $\mu < 0$; fixed point at the origin does not participate in saddle-node coalescence of cycles bifurcation.

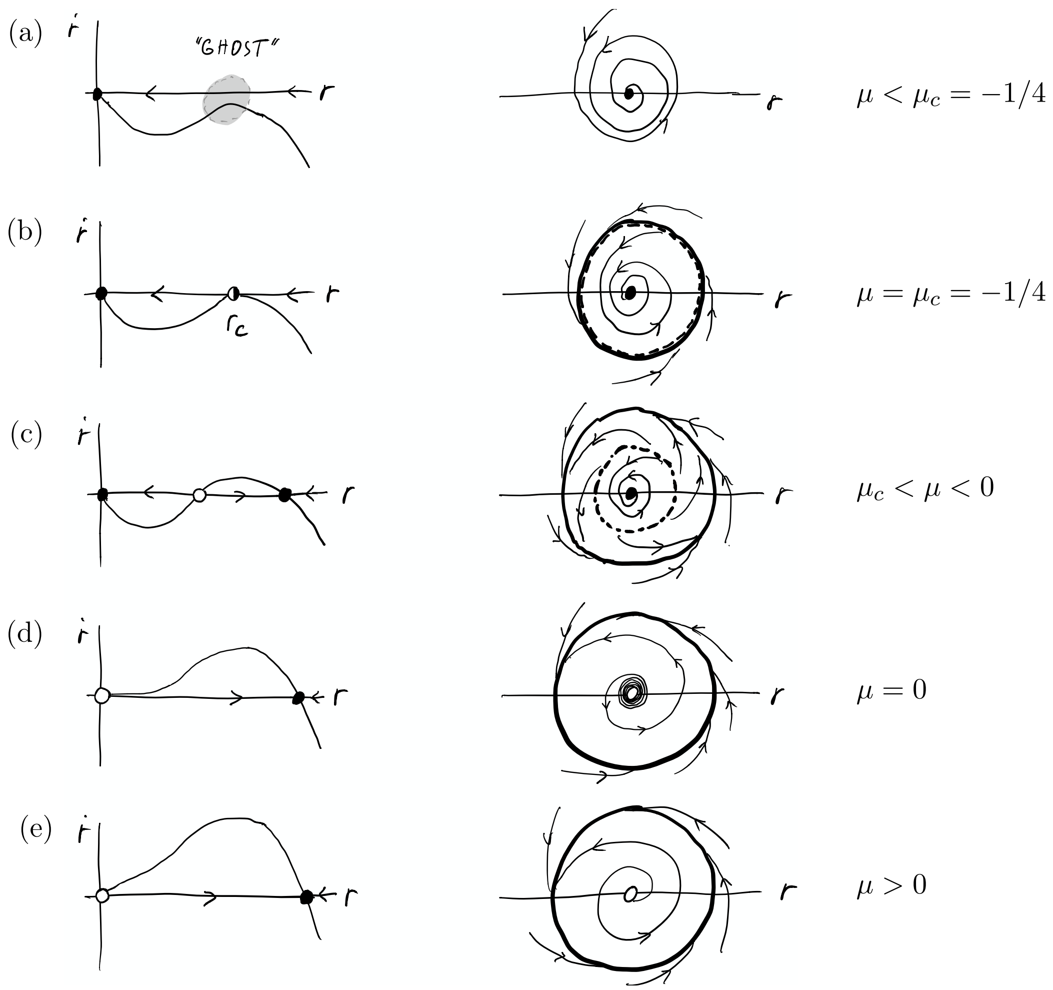
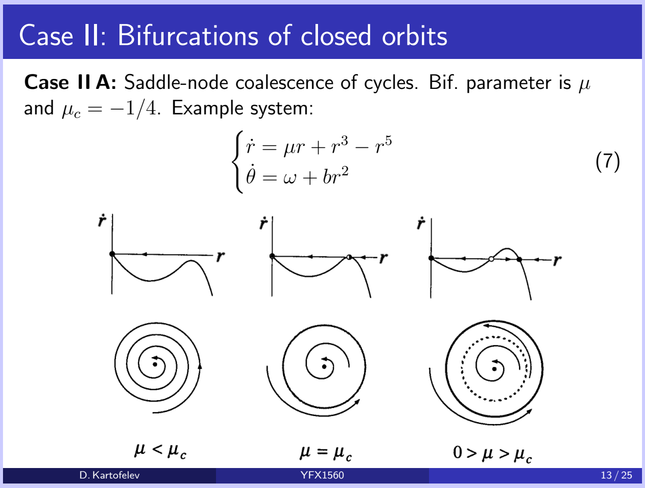


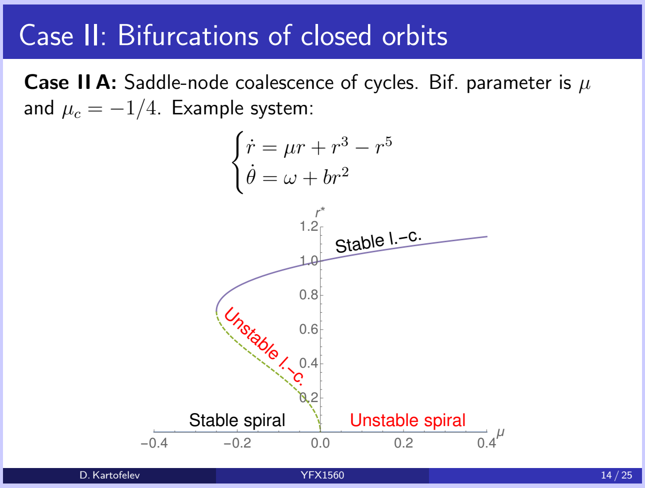
Figure 9: Dynamics in radial direction r and phase portraits for varied μ. (a) Case $\mu < \mu_c = -1/4$, featuring a "ghost" region with slowed down flow. The *ghost* is predicting the appearance of the saddle-node coalescence of cycles bifurcation. (b) Case $\mu = \mu_c = -1/4$, the saddle-node coalescence of cycles bifurcation point featuring a half-stable limit-cycle. (c) Case $\mu_c < \mu < 0$, featuring a high-amplitude stable limit-cycle and lower amplitude unstable limit-cycle. (d) Case $\mu = 0$, the subcritical Hopf bifurcation where the unstable limit-cycle has merged with the origin and changed the stability of the spiral, *cf.* Sec. 7. The close proximity of the spiral features the **algebraic** critically slowed flow dynamics. (e) Case $\mu > 0$, featuring a high-amplitude stable limit-cycle and the unstable spiral at the origin.

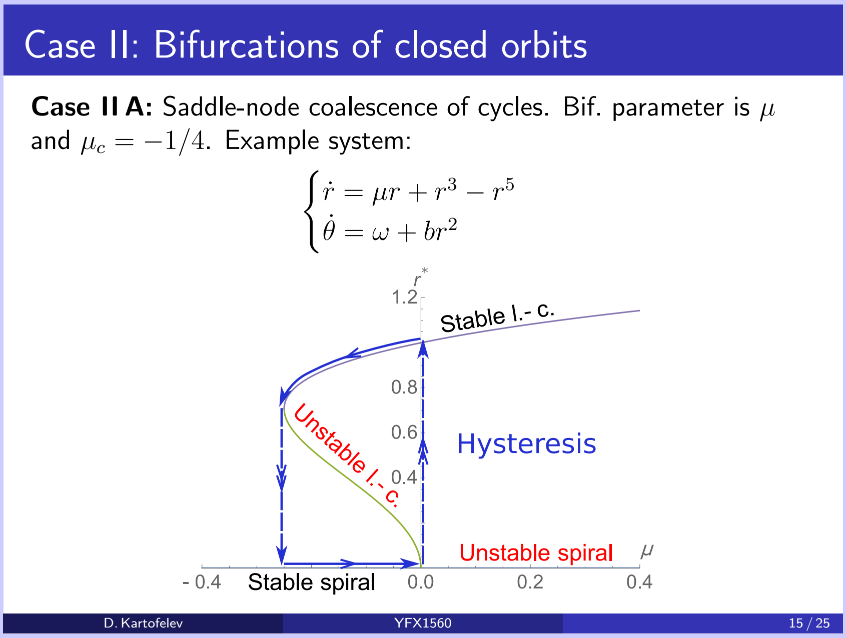
The presented results can be checked against numerical calculations. The dynamics in radial direction, time-domain solutions, and the phase portraits are shown in the following interactive numerical file.

NUMERICS: NB#5
Examples of bifurcations in 2-D systems: a saddle-node coalescence of limit-cycles. Bifurcation diagram.

SLIDES: 13–15







Slides 14 and 15 show the bifurcation diagram for saddle-node coalescence of cycles bifurcation. The subcritical Hopf bifurcation for $\mu = 0$ is responsible for **hysteresis on the level of cycles**, shown on Slide 15. Systems exhibiting this type of hysteresis are considered **dangerous**. Reverting the system to the stable fixed point located at the origin requires large changes in control parameter μ.

8.1 Why is the subcritical Hopf bifurcation considered *dangerous*?

As shown above and similarly in the case of the pitchfork bifurcations, the Hopf bifurcations come in both supercritical and subcritical varieties. The subcritical case is always **much more dramatic, and potentially dangerous** especially in engineering applications. After the bifurcation, the trajectories must jump to a distant attractor, which may be a fixed point, another limit-cycle (see Slide 15), infinity, or in three and higher dimensions a chaotic attractor (discussed in future lectures). The subcritical Hopf bifurcation with an accompanying attractor combined are sometimes referred to as the *Hopf* bifurcation.

An additional danger lies in the fact that linear analysis via linearisation can't distinguish the supercritical Hopf from subcritical one. The bifurcating features of systems featuring the Hopf bifurcation are always described by nonlinear terms. An analytical criterion for determining the subcritical Hopf exists, but it can be difficult to use and it's too complicated for our purposes. A quick and dirty approach is to use a computer. If a small, attracting limit cycle appears immediately after the fixed point goes unstable, and if its amplitude shrinks back to zero as the parameter is reversed, the bifurcation is supercritical; otherwise, it's probably subcritical, in which case the nearest attractor might be far from the fixed point, and the system may exhibit hysteresis as the parameter is reversed, see Slide 15. Of course, computer experiments are not proofs and you should check and understand numerical integration methods and algorithms most carefully before making any firm conclusions.

8.2 Examples of instabilities related to the subcritical Hopf bifurcation

SLIDES: 16–20

The subcritical Hopf bifurcations occur in the dynamics of nerve cells, in experimental chemistry of chemical oscillators, in aeroelastic flutter and other vibrations of airplane wings, and in instabilities of fluid flows. These systems are remarkable for their spectacular behaviours. Let's watch some of them in action:

9 CASE II B: SNIPER (saddle-node infinite period bifurcation)

An **example decoupled system** in polar coordinates is given in the form:

$$\begin{aligned} \dot{r} &= r\left(1 - r^2\right), \\ \dot{\theta} &= \omega - \sin\theta, \end{aligned} \tag{5}$$

where angular velocity parameter ω is the bifurcation parameter. This system combines two decoupled one-dimensional systems. Figure 10 shows the dynamics for varied ω. In radial direction r, all trajectories, except $r^* = 0$, approach the unit circle monotonically as $t \to \infty$. In the angular direction θ, the motion is everywhere counterclockwise if $\omega > 1$, whereas there are two invariant rays defined by

$$\dot{\theta} = 0 \quad \Rightarrow \quad \sin\theta = \omega, \tag{6}$$

existing for $\omega < 1$. Hence, as ω increases (or decreases) through the bifurcation point $\omega = \omega_c = 1$, the phase portraits changes as shown in Fig. 10, and on Slides 21 and 22.

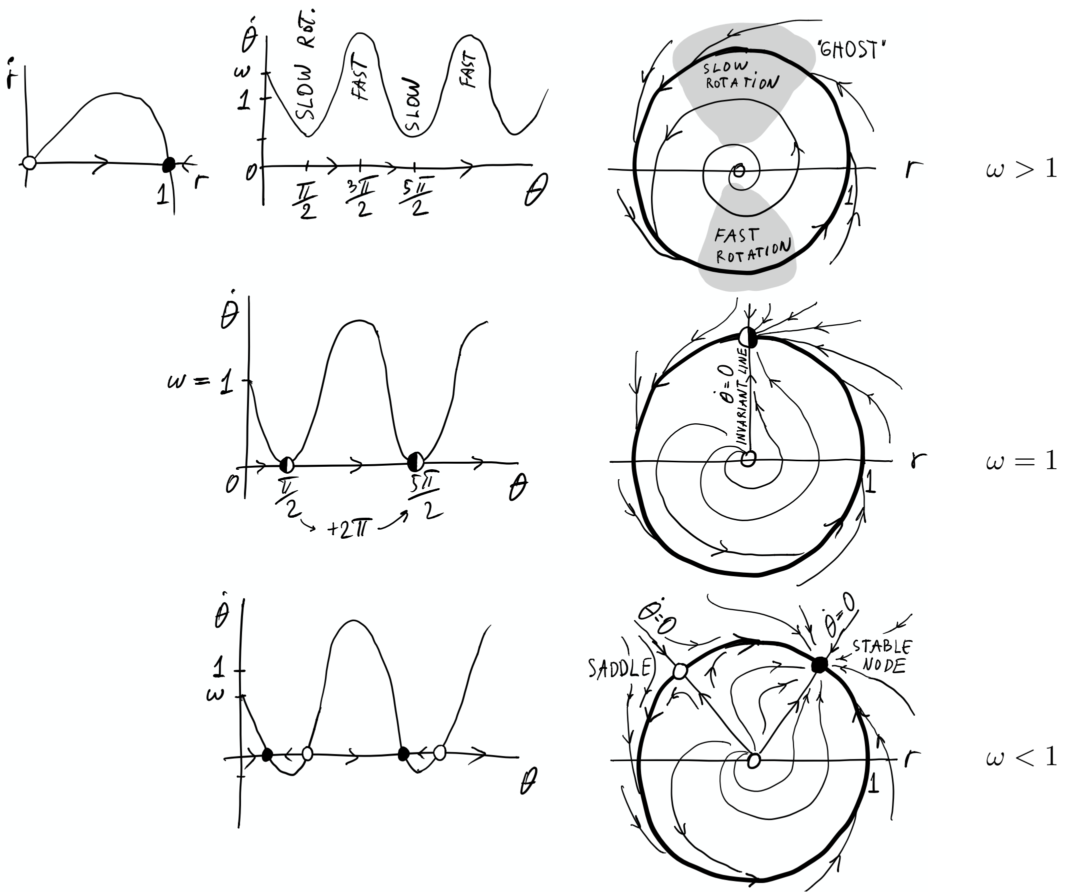
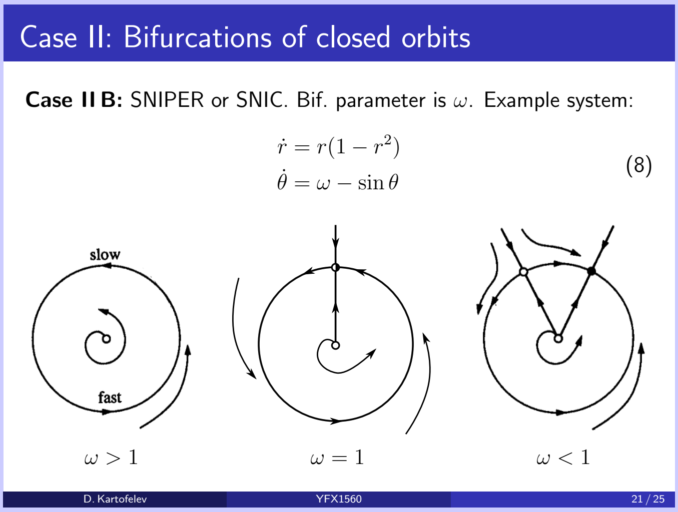
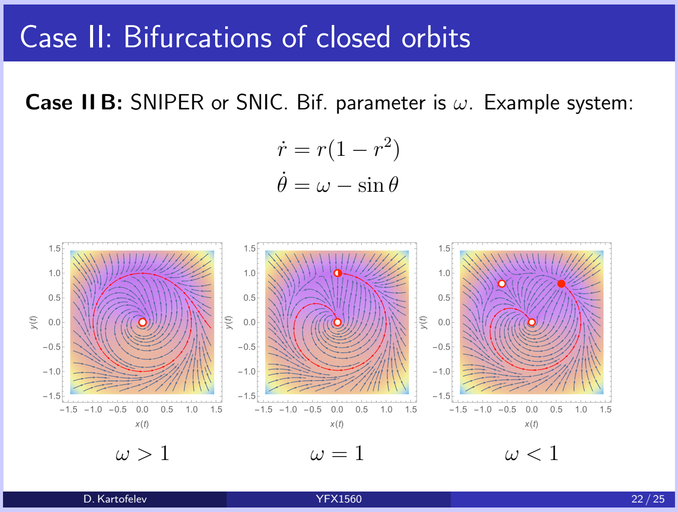


Figure 10: One-dimensional phase portraits of the first and second equations of Sys. (5) are shown above in the first two columns, and the resulting two-dimensional phase portrait is shown in the third column on the polar plane for varied parameter ω. (Top) Case $\omega \gtrsim 1$, featuring a slow flowing bottlenecking *ghost* region that is predicting the imminent appearance of a SNIPER bifurcation. (Middle) Case $\omega = 1$, the bifurcation point. (Bottom) Case $\omega < 1$, featuring two invariant sets or rays starting at the origin defined by $\dot{\theta} = 0$.

As $\omega \gtrapprox 1$ decreases, the limit-cycle located at $r = 1$ develops a bottlenecking *ghost* region at $\theta = \pi/2$ that becomes increasingly severe as $\omega \to 1^+$. The oscillation period lengthens and finally becomes infinite at $\omega = 1$, when a half-stable fixed point appears on the limit-cycle; hence the term **infinite period bifurcation**. For $\omega < 1$, the fixed point splits into a **saddle** and a **stable node**. The created fixed points remain on the limit-cycle given by $r = 1$.

SLIDES: 21, 22





The presented results can be checked against numerical calculations. The dynamics in radial and angular directions, time-domain solutions, and the phase portraits are shown in the following numerical file.

NUMERICS: NB#6

Examples of bifurcations in 2-D systems: SNIPER (saddle-node infinite period bifurcation). Integrated solution and phase portrait.

The same numerical file contains an example of SNIPER bifurcation happening in a simplified and idealised model of mammalian electrocardiogram signal (EKG).

NUMERICS: NB#6

Examples of bifurcations in 2-D systems: SNIPER (saddle-node infinite period bifurcation). Integrated solution and phase portrait.

Example: Destruction of the closed orbit in a signal similar to human electrocardiogram (ECG).

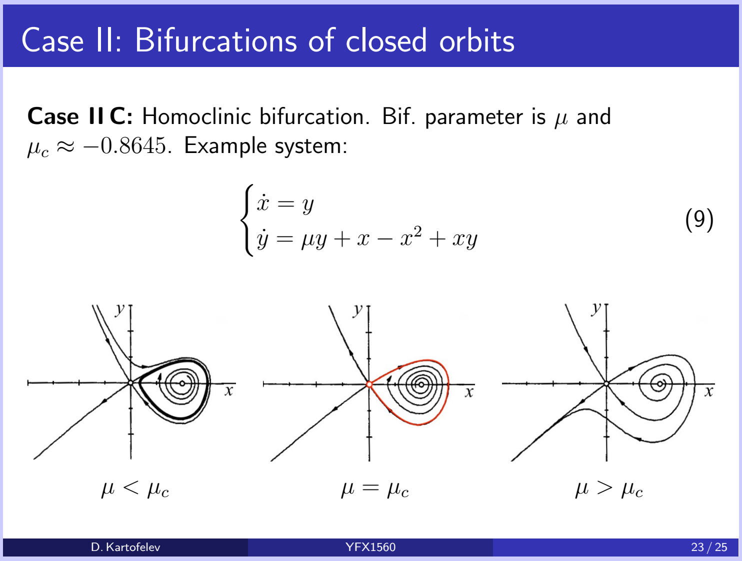
10 CASE II C: Homoclinic bifurcation or saddle-loop bifurcation

In this scenario, part of a limit-cycle moves closer and closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a **homoclinic orbit**. This is another kind of **infinite period bifurcation**; to avoid confusion, we'll call it a **saddle-loop or homoclinic bifurcation**.

Phase portraits of a solitary wave or a **soliton** wave propagation models feature this type of homoclinic bifurcation. The homoclinic orbit at the bifurcation point corresponds to a self-reinforcing and stable solitary wave.

SLIDE: 23

Slide 23 presents an **example system** for homoclinic bifurcation:



For $\mu < \mu_c$ a stable limit-cycle, shown with the bold curve on Slide 23, is located close to the saddle point. As μ increases to $\mu = \mu_c$ the limit-cycle swells and bangs into the saddle resulting in the merger of the two, creating a **homoclinic orbit**. The **homoclinic orbit** present at the bifurcation point $\mu = \mu_c \approx -0.8645$ is shown with the red trajectory. Once $\mu > \mu_c$ the saddle connection breaks and the orbit is destroyed.

The presented results can be checked against numerical calculations. The time-domain solution and the phase portraits are shown in the following interactive numerical file.

NUMERICS: NB#7

Examples of bifurcations in 2-D systems: a homoclinic bifurcation or saddle-loop bifurcation (solitons). Integrated solution and phase portrait.

Revision questions

1. Classification of bifurcations in 2-D.
2. What is the Hopf bifurcation?
3. What is the supercritical Hopf bifurcation?
4. What is the subcritical Hopf bifurcation?
5. What are global bifurcations of closed orbits?
6. Name some global bifurcations of closed limit-cycles.
7. What is a saddle-node coalescence (or bifurcation) of limit-cycles?
8. What is hysteresis on level of cycles?
9. Name dangers associated with the *Hopf* bifurcation.
10. What is a saddle-node infinite period bifurcation?
11. What is a (saddle-loop or) homoclinic bifurcation?
12. Name examples of dynamical instabilities.