Lecture 4: 2-D homogeneous linear systems, classification of fixed points in 2-D systems, the Lyapunov stability, basin of attraction

Contents

Handout: Classification of fixed points in linear homogeneous 2-D systems.

1 Linear homogeneous 2-D systems

1.1 Introduction

We continue our discussion on the second-order or 2-D systems in the form:

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \tag{1}$$

where $\vec{f} = (f(x,y),\, g(x,y))^T$ is a vector containing arbitrary given functions f and g, $\vec{x} = (x,y)^T$, $\vec{x} \in \mathbb{R}^2$ is the free variable vector. The component form of Eq. (1) is the following:

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y). \end{cases} \tag{2}$$

A **fixed point** of system (1) is defined as follows:

$$\dot{\vec{x}} = 0 \quad \Rightarrow \quad \vec{f}(\vec{x}^*) = \vec{0}, \tag{3}$$

and in the component form (2) we write:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} f(x^*,y^*) = 0, \\ g(x^*,y^*) = 0. \end{cases} \tag{4}$$

Solving (3) or (4) for $\vec{x}^*$ or x^* and y^* will give coordinate values of the fixed point $\vec{x}^* = (x^*,y^*)^T$.

In this lecture we'll discuss second-order linear **homogeneous** systems with **constant coefficients**. The following slide shows an example of a homogeneous differential equation and compares it to its **nonhomogeneous** and **non-autonomous** counterparts.

Slide: 3

Linear second-order differential equations

- Homogeneous differential equation:

$$a\ddot{x} + b\dot{x} + cx = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x} = y \\ a\dot{y} = -by - cx \end{cases} \tag{1}$$

- Nonhomogeneous differential equation:

$$a\ddot{x} + b\dot{x} + cx = f(x) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = y \\ a\dot{y} = -by - cx + f(x) \end{cases} \tag{2}$$

- Non-autonomous differential equation:

$$a\ddot{x} + b\dot{x} + cx = g(t) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = y \\ a\dot{y} = -by - cx + g(t) \end{cases} \tag{3}$$

In cases (1) and (2) a, b, and c are the constant coefficients. In case (3) a, b, and c may depend on time t but don't have to. Above functions f and g are arbitrarily selected linear functions.

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The notion of non-autonomous systems was mentioned in Lecture 1.

1.2 2-D phase portrait plotting

As introduced in the previous lecture, a phase portrait of second-order Sys. (2) is constructed by plotting vector field $\dot{\vec{x}} = (\dot{x},\dot{y})^T = (f(x,y), g(x,y))^T$ against independent variables x and y in xy-plane, as shown in Fig. 1 and on Slide 4. A **trajectory** within this vector field represents a single solution of Sys. (2) corresponding to an initial condition—a single point in the plane (x_0, y_0), where $x_0 = x(0)$ and $y_0 = y(0)$.

Essence of the course (idea by Henri Poincaré): Construction of phase portrait allows us to find all qualitatively different trajectories or solutions of a system without solving the system itself explicitly (analytically). In part, we already have demonstrated that during Lecture 1 in the 1-D case.

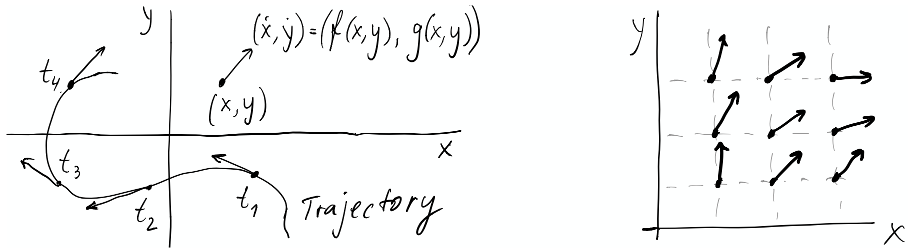
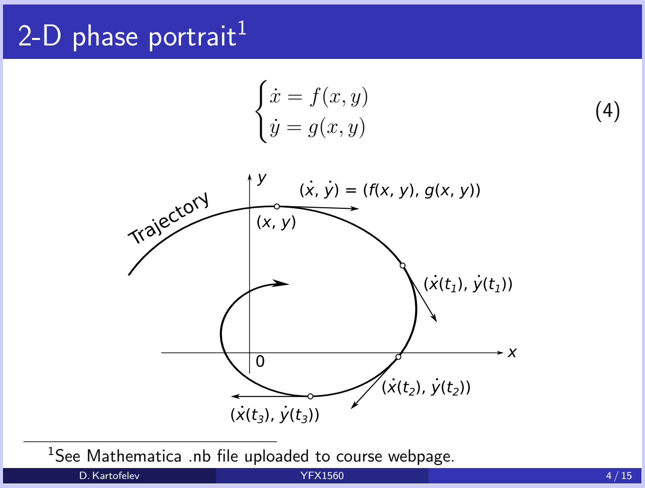


Figure 1: (Left) Phase portrait of a 2-D or a second-order system. A trajectory is shown with the continuous curve where time moments $t = t_1 < t_2 < t_3 < t_4$. (Right) A simple idea behind the construction of a 2-D phase portrait—an entire vector field for the shown ranges of variables x and y. Plotting entire vector field for the shown ranges of variables x and y. The vector field vectors are evaluated and shown at points (x, y) that are placed in a uniform grid. This grid is shown with the dashed lines.

SLIDE: 4



An example of phase portrait plotting procedure and different visualisation styles of phase portraits are shown in the following numerical file.

NUMERICS: NB#1

Vector field plotting. Integrated solution and phase portrait of the damped harmonic oscillator.

1.3 Why bother with linear homogeneous 2-D systems?

In nonlinear systems it is harder to determine the stability and type of fixed points, due to a possibility of more complex dynamics present in the phase portraits. For this reason, in future lectures, we will be **linearising nonlinear systems** and then studying the stability and dynamics of the corresponding linearised systems with the aim to gain insight into the original nonlinear systems. The conditions under which this approach is allowed and feasible will be discussed in the lectures to come. This means that we need to familiarise ourselves with linear second-order systems and their dynamics.

The general form of linear homogeneous system is the following:

$$\begin{cases} \dot{x} = ax + by, \\ \dot{y} = cx + dy, \end{cases} \tag{5}$$

where $\{a, b, c, d\} \in \mathbb{R}$ are the **constant coefficients**. By defining $\vec{x} = (x, y)^T$ we rewrite Sys. (5) in the matrix form:

$$\dot{\vec{x}} = A\vec{x} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}, \tag{6}$$

where matrix A is the system or coefficient matrix of Sys. (5). Dynamics of linear systems is fully determined by the system matrix A eigenvalues λ_i and eigenvectors $\vec{v}_i$ where $i = 1, 2$ for 2-D systems.

1.4 Examples

1.4.1 Harmonic oscillator

Harmonic oscillator is described by the linear system in the following form:

$$\ddot{x} + x = 0 \quad \Rightarrow \quad \begin{bmatrix} \text{assuming:} \\ y = \dot{x} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \dot{x} = y, \\ \dot{y} = -x, \end{cases} \tag{7}$$

where x is the displacement. The matrix form of this system for $\vec{x} = (x, y)^T$ and appropriately selected system matrix A is

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}. \tag{8}$$

A solution and phase portrait of harmonic oscillation (7) are shown in the following numerical file.

NUMERICS: NB#1

Vector field plotting. Integrated solution and phase portrait of the damped harmonic oscillator.

1.4.2 Damped harmonic oscillator

Damped harmonic oscillator is described by the linear system in the following form:

$$\ddot{x} + x \underbrace{+\,\dot{x}}_{\text{damping}} = 0 \quad \Rightarrow \quad \begin{bmatrix} \text{assuming:} \\ y = \dot{x} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \dot{x} = y, \\ \dot{y} = -x - y, \end{cases} \tag{9}$$

where x is the displacement and the last term on the right-hand side of the equation is the attenuation or damping or friction term. The matrix form of this system for $\vec{x} = (x, y)^T$ and appropriately selected system matrix A is the following:

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}. \tag{10}$$

A solution and phase portrait of damped harmonic oscillation (9) are shown in the following file.

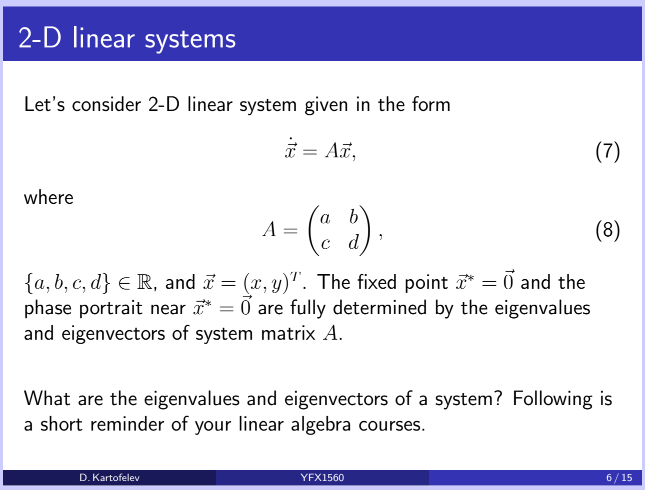
NUMERICS: NB#1

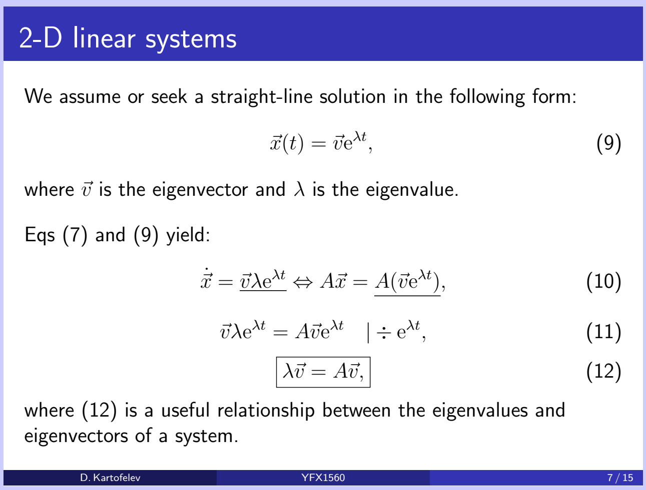
Vector field plotting. Integrated solution and phase portrait of the damped harmonic oscillator.

2 Eigenanalysis of 2-D linear systems

As stated above, fixed point dynamics of a linear 2-D system is determined by an **eigenanalysis of its system matrix** A. The following is a short reminder of your linear algebra courses.

SLIDES: 6–9





2-D linear systems

The straight-line solution exists if one can find eigenvalues λ_i and eigenvectors $\vec{v}_i$.

Eigenvalue λ is given by

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \boxed{\lambda^2-\tau\lambda+\Delta=0}, \quad (13)$$

where the boxed in part is called the characteristic equation of a system, and where

$$\tau = a+d \quad \text{is the trace of matrix } A\text{, and} \quad (14)$$

$$\Delta = ad-bc \quad \text{is the determinant of matrix } A. \quad (15)$$

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2-D linear systems

$$\boxed{\lambda^2-\tau\lambda+\Delta=0}$$

The algebraic form of two eigenvalues λ_i is the following:

$$\lambda_{1,2}=\frac{\tau\pm\sqrt{\tau^2-4\Delta}}{2}. \quad (16)$$

Additional *nice* properties of λ_i and Δ are the following:

$$\tau=\lambda_1+\lambda_2, \quad (17)$$

$$\Delta=\lambda_1\lambda_2. \quad (18)$$

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Dynamics of linear 2-D systems is determined by system matrix determinant Δ and trace τ.

One can use a computer or even an Internet search engine to perform eigenanalysis of square matrices. The following numerical file contains an example of eigenanalysis.

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

3 Classification of fixed points in 2-D linear systems

All possible behaviours found in *linear* homogeneous second-order systems are presented here. The classification of fixed points is presented in terms of system matrix determinant Δ and trace τ.

3.1 CASE 1: Saddle node, saddle point or saddle

Criterion for determining the fixed point:

$$\boxed{\Delta < 0.} \quad (11)$$

This criterion holds for real and distinct (not equal, different) eigenvalues and linearly independent (not on the same line) eigenvectors. Since $\Delta=\lambda_1\lambda_2$, we must have $\lambda_1 \lessgtr 0$ and $\lambda_2 \gtrless 0$. The determination of saddle node fixed point does not depend on trace τ, $(-\infty<\tau<\infty)$. Figures 2 and 3 show the dynamics of a saddle node.

Stability: The fixed point $\vec{x}^*=\vec{0}=(0,0)^T$ is always unstable, although the phase portrait has one stable eigendirection (the other one is always unstable).

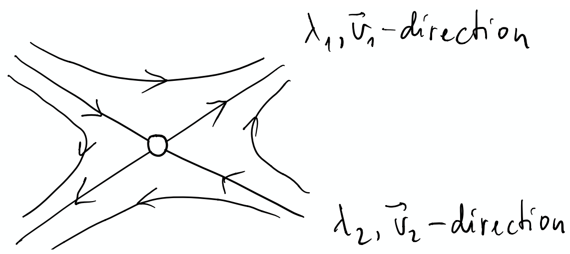


Figure 2: Phase portrait of a saddle node fixed point, where eigenvalues $\lambda_1>0$ and $\lambda_2<0$.

General solution of the system:

$$\vec{x}(t)=C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2, \quad (12)$$

where C_i are the integration constants and $\vec{v}_i$ are the eigenvectors. See Figs 2 and 3.

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 2.

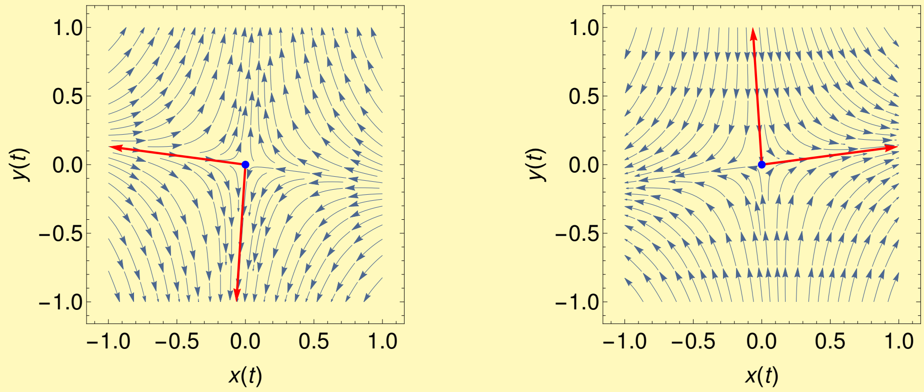


Figure 3: Phase portrait and eigenvectors. Eigenvectors are shown with the red arrows. (Left) Saddle node. (Right) Saddle node, different main flow directions.

3.2 CASE 2a: Node

Criterion for determining the fixed point:

$$\boxed{\Delta > 0} \quad \text{and} \quad \boxed{\tau^2 - 4\Delta > 0.} \tag{13}$$

This criterion holds for real and distinct eigenvalues having the same sign, $\lambda_1, \lambda_2 \gtrless 0$. Eigenvectors are linearly independent. The criterion $\tau^2 - 4\Delta > 0$ simply means that we are located outside the region defined by $\tau^2 - 4\Delta = 0$ (or the region defined by $\tau > \pm 2\sqrt{\Delta}$ and $\Delta > 0$) shown in the overview plot in Fig. 18 and on Slide 10. Figures 4 and 5 show the dynamics of a node.

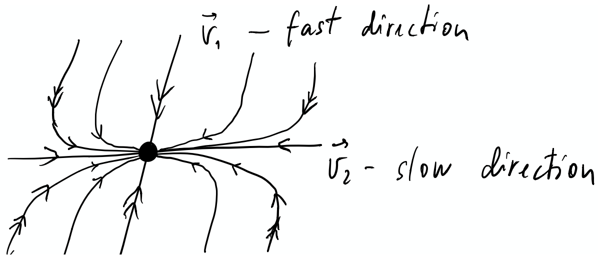


Figure 4: Phase portrait of a stable node where $\lambda_1 < \lambda_2 < 0$. As $t \to \infty$ trajectories approach the fixed point tangentially to the slower eigendirection.

Stability: The fixed point $\vec{x}^* = \vec{0}$ may be either stable (attracting sink) or unstable (repelling source). If

$$\boxed{\tau < 0,} \tag{14}$$

then we have a stable node, and from (13) it also follows that $\lambda_1, \lambda_2 < 0$. If

$$\boxed{\tau > 0,} \tag{15}$$

then we have an unstable node, and from (13) it follows that $\lambda_1, \lambda_2 > 0$.

General solution of the system:

$$\vec{x}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2, \tag{16}$$

where C_i are the integration constants and $\vec{v}_i$ are the eigenvectors. See Figs 4 and 5.

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 4.

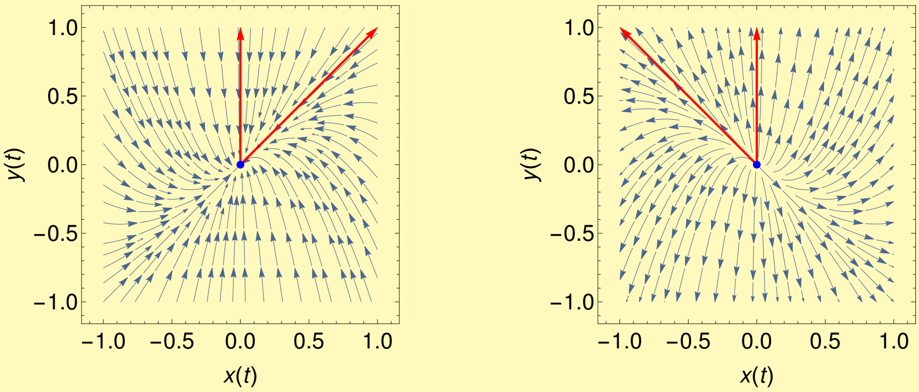


Figure 5: Phase portrait and eigenvectors. Eigenvectors are shown with the red arrows. (Left) Stable node. (Right) Unstable node.

3.3 CASE 2b: Spiral

Criterion for determining the fixed point:

$$\boxed{\Delta > 0} \quad \text{and} \quad \boxed{\tau^2 - 4\Delta < 0.} \tag{17}$$

This criterion holds for complex and distinct eigenvalues that are complex conjugates of each other. The criterion $\tau^2 - 4\Delta < 0$ simply means that we are located inside the region defined by $\tau^2 - 4\Delta = 0$ (or the region defined by $\tau < \pm 2\sqrt{\Delta}$ and $\Delta > 0$ that excludes Δ-axis) shown in the overview plot in Fig. 18 and on Slide 10. Figures 6 and 7 show the dynamics of a spiral.

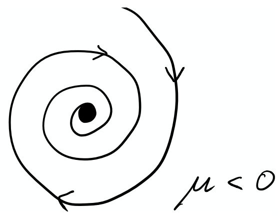


Figure 6: Phase portrait of a stable spiral with complex eigenvalues $\lambda_\pm = \mu \pm \mathrm{i}\omega$ where $\mu < 0$.

Physical interpretation of complex eigenvalues $\lambda_\pm = \mu \pm \mathrm{i}\omega$ is the following: real parts μ are related to decay rate ($\mu < 0$) of the spiral, and imaginary parts ω are related to the rotation rate.

Stability: The fixed point $\vec{x}^* = \vec{0}$ may be either stable (attracting sink) or unstable (repelling source). If

$$\boxed{\tau < 0,} \tag{18}$$

witch also implies that $\mu < 0$, then we have a stable spiral. If

$$\boxed{\tau > 0,} \tag{19}$$

witch also implies that $\mu > 0$, then we have an unstable spiral.

General solution of the system: A linear combination of

$$\mathrm{e}^{\mu t} \cos \omega t, \tag{20}$$

and

$$\mathrm{e}^{\mu t} \sin \omega t. \tag{21}$$

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 6.

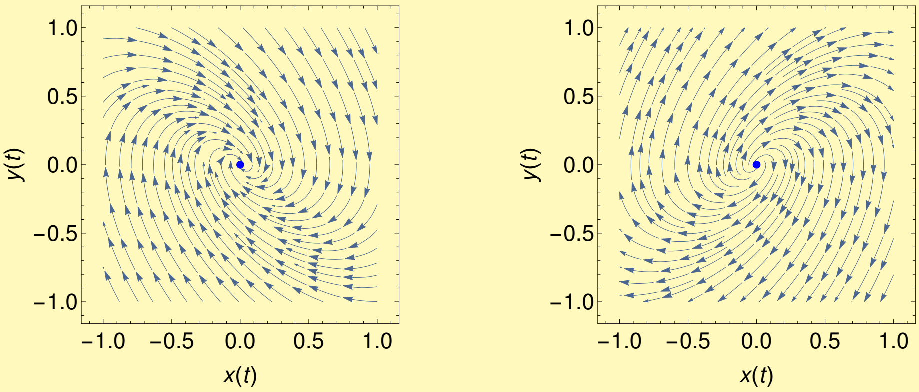


Figure 7: Phase portraits of two spirals with complex eigenvalues $\lambda_\pm = \mu \pm \mathrm{i}\omega$. (Left) Stable spiral where $\mu < 1$. (Right) Unstable spiral where $\mu > 1$.

3.4 CASE 3: Center

Criterion for determining the fixed point:

$$\boxed{\Delta > 0} \quad \text{and} \quad \boxed{\tau = 0.} \tag{22}$$

This criterion holds for pure imaginary eigenvalues $\lambda_\pm = \pm \mathrm{i}\omega$ (complex conjugate pair). Figures 8 and 9 show the dynamics of a center.

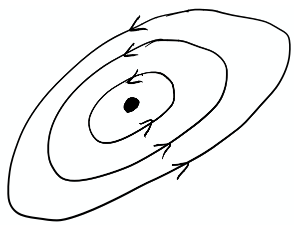


Figure 8: Phase portrait of a Lyapunov stable center.

Stability: The fixed point $\vec{x}^* = \vec{0}$ is said to be Lyapunov stable. The 2-D systems can feature a new type of stability called the Lyapunov stability. A fixed point is said to be **Lyapunov stable** if trajectories starting *near* a fixed point stay *near* it for all time not just for $t \to \infty$ (note: a more rigorous definition exists). The centers are Lyapunov stable because the real part of its eigenvalues $\mu = 0$ and thus the decay rate of the rotation is absent.

General solution of the system: A linear combination of

$$\cos \omega t, \tag{23}$$

and

$$\sin \omega t. \tag{24}$$

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 8.

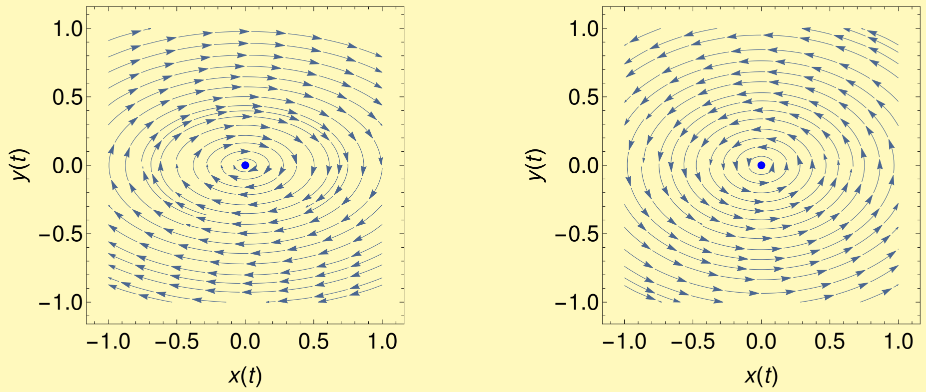


Figure 9: Phase portrait. (Left) Center featuring clockwise rotation. (Right) Center featuring counter-clockwise rotation.

Which way is vector field rotating? A spiral or center can rotate in clockwise or counter-clockwise directions. This question is not an issue when one uses a computer to construct the phase portrait. But, let's say you need to sketch the portrait by hand. System matrix A does not explicitly give you the direction. In order to determine the direction simply calculate one vector and the direction of the entire flow becomes obvious.

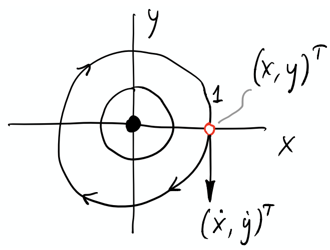


Figure 10: Determination of the rotation direction. The flow vector, calculated at coordinates $(x, y)^T = (1, 0)^T$, is shown with the arrow. The phase portrait features a Lyapunov stable fixed point at its origin.

Example: We consider a simple case that was introduced above—harmonic oscillator defined by Eq. (7):

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases}$$

Let's say we are interested in a vector located at $\vec{x} = (x, y)^T = (1, 0)^T$. This vector is shown with the red bullet in Fig. 10. The resulting vector is

$$\begin{cases} \dot{x} = y = 0 \\ \dot{y} = -x = -1 \end{cases} \quad \Rightarrow \quad \dot{\vec{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{25}$$

This vector is shown on the phase portrait in Fig. 10. From here it is clear that vector field is rotating in a clockwise direction. The fixed point of harmonic oscillator shown in Fig. 10 is Lyapunov stable.

3.5 CASE 4a: Degenerate node of the first type

Criterion for determining the fixed point:

$$\boxed{\Delta > 0} \quad \text{and} \quad \boxed{\tau^2 - 4\Delta = 0.} \tag{26}$$

This criterion holds for real and repeated (equal) eigenvalues and for one uniquely determined eigenvector (the other one can be anything). The criterion $\tau^2 - 4\Delta = 0$ simply means that we are located on the line defined by $\tau^2 - 4\Delta = 0$ shown in overview plot in Fig. 18 and on Slide 10. Figures 11 and 13 show the dynamics of this fixed point.

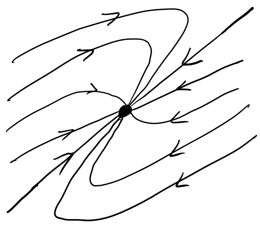


Figure 11: Phase portrait of a stable degenerate node of the first type.

Figure 12: (Left) Stable degenerate node as a failed stable spiral. The trajectories of failed spiral are shown with the blue dashed curves. (Right) Characteristics of a node. The blue coloured trajectories can be mistaken for a node.

Stability: The fixed point $\vec{x}^* = \vec{0}$ can be either stable or unstable. If

$$\boxed{\tau < 0,} \tag{27}$$

which also implies $\lambda_1 = \lambda_2 < 0$, then the fixed point is stable. If

$$\boxed{\tau > 0,} \tag{28}$$

which also implies $\lambda_1 = \lambda_2 > 0$, then the fixed point is unstable.

General solution: Omitted from this document.

Degenerate nodes are borderline cases bordering spirals and nodes, see the overview plot shown in Fig. 18 and on Slide 10. In a certain sense they posses characteristics of both. Figure 12 shows the spiral and nodal characteristics hidden within a degenerate node.

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 11.

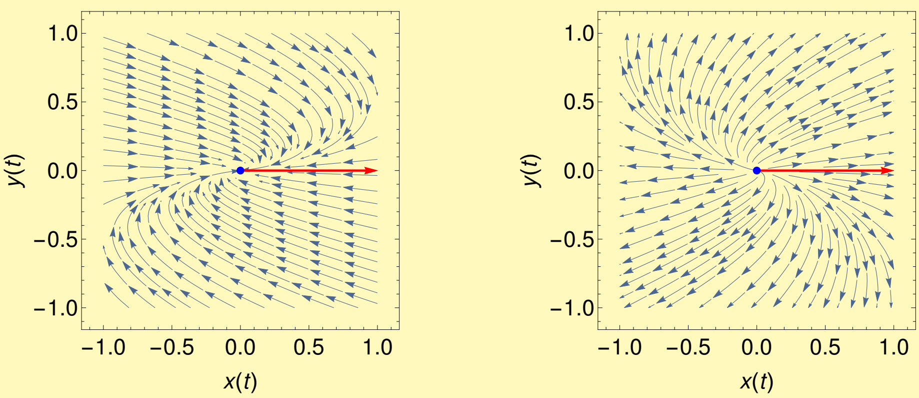


Figure 13: Phase portrait and a non-unique eigenvector (unique one is coincidentally $|\vec{v}| = 0$). This eigenvector is shown with the red arrow. (Left) Stable degenerate node of the first type. (Right) Unstable degenerate node of the first type.

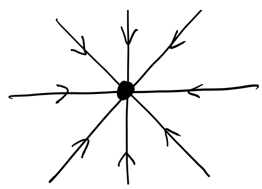


Figure 14: Phase portrait of a stable degenerate node of the second type or simply a stable star.

3.6 CASE 4b: Degenerate node of the second type or star

Criterion for determining the fixed point:

$$\boxed{\Delta > 0} \quad \text{and} \quad \boxed{\tau^2 - 4\Delta = 0.} \tag{29}$$

This criterion holds for real and repeated (equal) eigenvalues and for not uniquely determined eigenvectors (both can be anything, every direction is an eigendirection). The criterion $\tau^2 - 4\Delta = 0$ simply means that we are located on the curve defined by $\tau^2 - 4\Delta = 0$ (or on the curve defined by $\tau = \pm 2\sqrt{\Delta}$ and $\Delta > 0$) shown in the overview plot in Fig. 18 and on Slide 10. Figures 14 and 15 show the dynamics of this fixed point.
Stability: The fixed point $\vec{x}^* = \vec{0}$ can be either stable or unstable. If

$$\boxed{\tau < 0,} \tag{30}$$

which also implies $\lambda_1 = \lambda_2 < 0$, then the fixed point is stable. If

$$\boxed{\tau > 0,} \tag{31}$$

which also implies $\lambda_1 = \lambda_2 > 0$, then the fixed point is unstable.
General solution: Omitted from this document.

Numerics: nb#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 14.

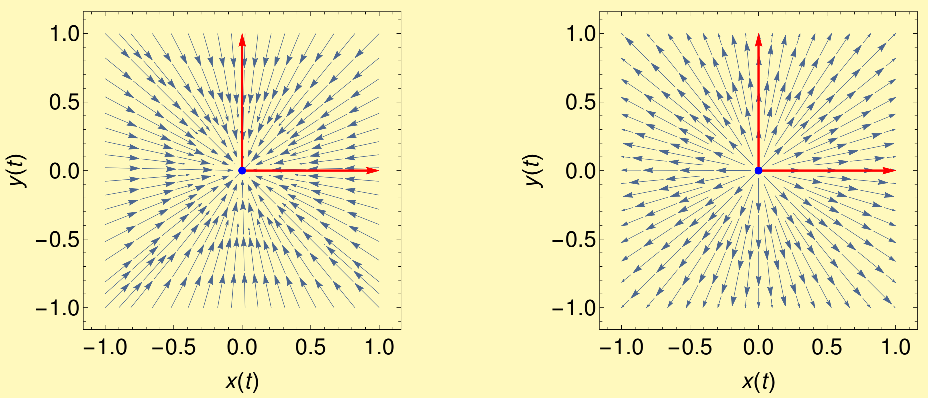


Figure 15: Phase portrait and non-unique eigenvectors. The eigenvectors are shown with the red arrows. (Left) Stable degenerate node of the second type or simply stable star. (Right) Unstable degenerate node of the second type or simply unstable star.

3.7 CASE 5a: Non-isolated fixed point, a line of fixed points

Criterion for determining the fixed point:

$$\boxed{\Delta = 0} \quad \text{and} \quad \boxed{\tau \neq 0.} \tag{32}$$

Figure 16: Phase portrait of non-isolated fixed points. (Left) Stable non-isolated fixed point. (Right) Unstable non-isolated fixed point.

This criterion holds for real eigenvalues. On the overview plot, shown in Fig. 18 and on Slide 10, we are located on the vertical τ-axis. Figures 16 and 17 show the dynamics of two non-isolated fixed points.
Stability: The fixed points (a line) can be either stable or unstable. If

$$\boxed{\tau < 0,} \tag{33}$$

then the fixed point is stable. If

$$\boxed{\tau > 0,} \tag{34}$$

then the fixed point is unstable.
General solution of the system:

$$\vec{x}(t) = C_1 \mathrm{e}^{\lambda_1 t} \vec{v}_1 + C_2 \mathrm{e}^{\lambda_2 t} \vec{v}_2, \tag{35}$$

where C_i are the integration constants and $\vec{v}_i$ are the eigenvectors. See Figs 16 and 17.

NUMERICS: NB#2

Calculation of matrix determinant, trace, eigenvalues and eigenvectors using computer. Classification of fixed points in 2-D linear homogeneous systems.

An example of quantitatively accurate phase portrait of the dynamics shown in Fig. 16.

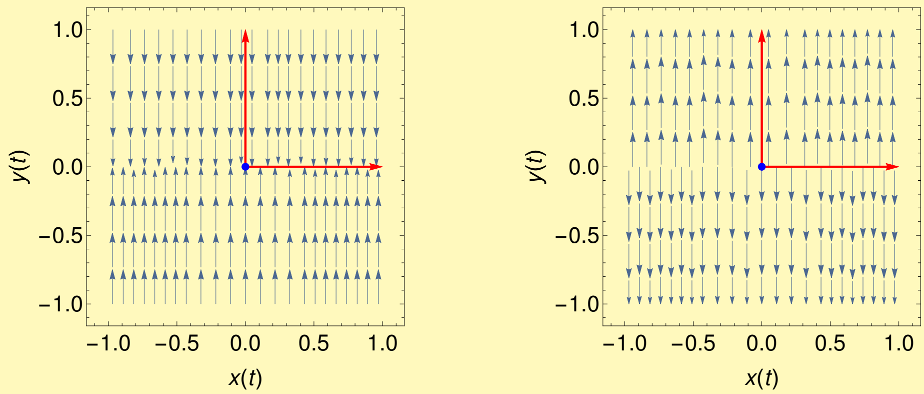


Figure 17: Phase portrait and eigenvectors. The eigenvectors are shown with the red arrows. (Left) Stable non-isolated fixed point. (Right) Unstable non-isolated fixed point.

3.8 CASE 5b: Non-isolated fixed point, a plane of fixed points

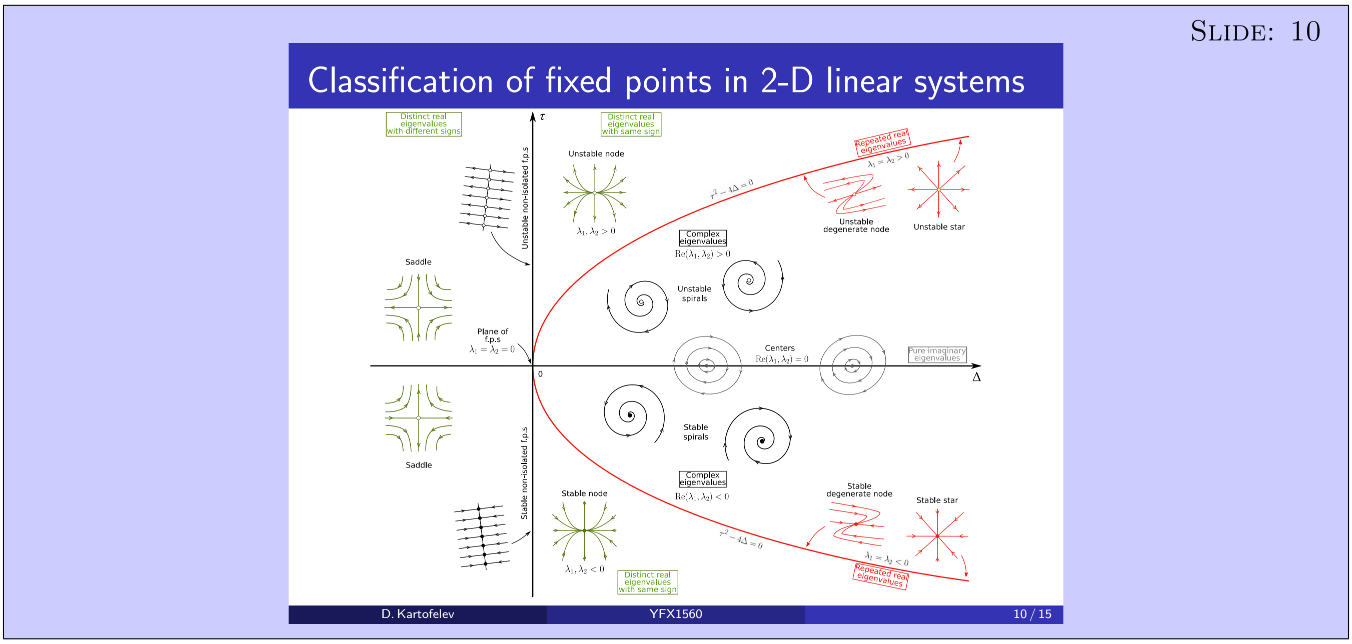
Criterion for determining the fixed point:

$$\boxed{\Delta = 0} \quad \text{and} \quad \boxed{\tau = 0.} \tag{36}$$

All points on the plane are fixed points. Nothing happens and nothing can happen! On the overview plot, shown in Fig. 18 and on Slide 10, criterion (36) is located at the origin of the graph.

3.9 Summary overview

As already alluded to, a nice and concise way of summarising the classification discussed in this lecture is shown on Slide 10 and in Fig. 18.



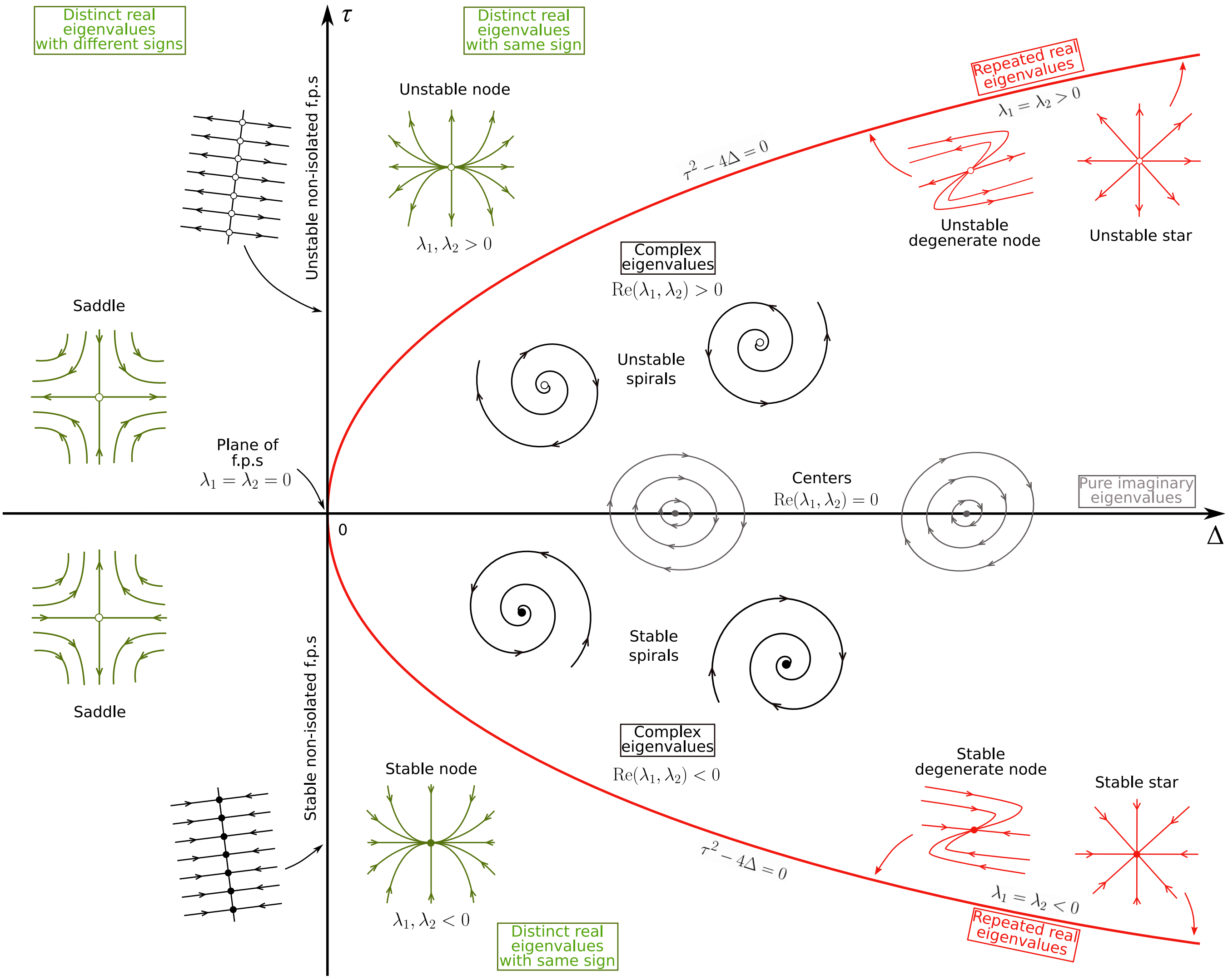


Figure 18: Classification of fixed points of linear homogeneous 2-D systems defined by $\dot{\vec{x}} = A\vec{x}$ where trace $\tau = \lambda_1 + \lambda_2$ and determinant $\Delta = \lambda_1 \lambda_2$ are determined by the 2×2 system matrix A.

Classification graph and chart presented above can also be summarised by the following flowchart:

- if $\Delta < 0$:
  Isolated fixed point
  CASE 1: **Saddle node**
- if $\Delta = 0$:
  Non-isolated fixed points
  - if $\tau < 0$:
    CASE 5a: **Line of stable fixed points**
  - if $\tau = 0$:
    CASE 5b: **Plane of fixed points**
  - if $\tau > 0$:
    CASE 5a: **Line of unstable fixed points**
- if $\Delta > 0$:
  Isolated fixed point
  - if $\tau < -\sqrt{4\Delta}$:
    CASE 2a: **Stable node**
  - if $\tau = -\sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
      CASE 4a: **Stable degenerate node**
    - if there are no uniquely determined eigenvectors (both are non-unique):
      CASE 4b: **Stable star**
  - if $-\sqrt{4\Delta} < \tau < 0$:
    CASE 2b: **Stable spiral**
  - if $\tau = 0$:
    CASE 3: **Center**
  - if $0 < \tau < \sqrt{4\Delta}$:
    CASE 2b: **Unstable spiral**
  - if $\tau = \sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
      CASE 4a: **Unstable degenerate node**
    - if there are no uniquely determined eigenvectors (both are non-unique):
      CASE 4b: **Unstable star**
  - if $\sqrt{4\Delta} < \tau$:
    CASE 2a: **Unstable node**

3.10 Note on degenerate nodes

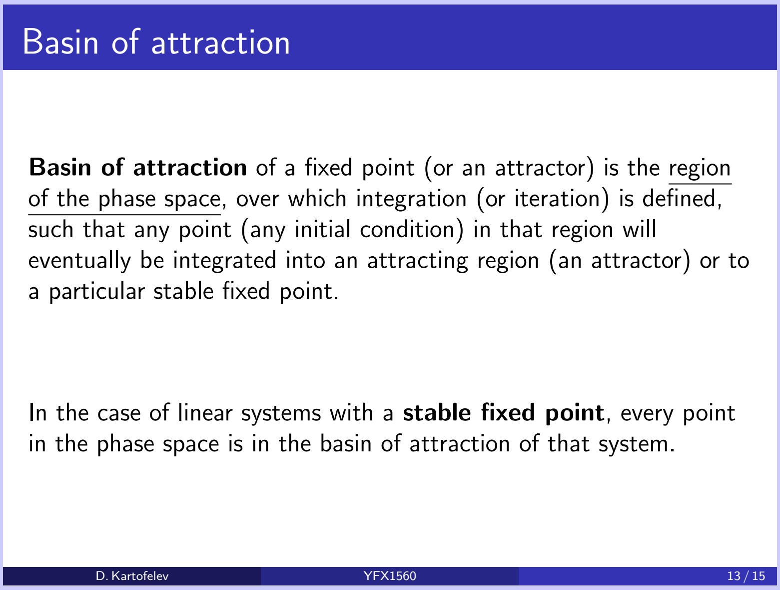
Figure 19: Comparison of a solution time-series of critically damped system that is shown on the left, to a solution time-series of damped system that is shown on the right.

The degenerate nodes of both types rarely occur in engineering applications. In stable cases or in forward-time, they represent critically damped or over-damped systems. We studied such a system in the previous lecture when we performed a bifurcation analysis of the dynamics of over-damped bead on a rotating hoop. These second-order systems achieve equilibrium without being *allowed* to oscillate. Figure 19 compares a possible solution of a critically damped oscillator to a solution of damped oscillator.

4 Basin of attraction

The definition of a basin of attraction is given on Slide 13. The notion of basin of attraction will be used and expanded upon in future lectures.

SLIDE: 13



The notions of iteration and attractor mentioned within the brackets will be further explained in future lectures.

Revision questions

1. How to plot a 2-D phase portrait of a system?
2. What are 2-D homogeneous linear systems?
3. What are non-homogeneous systems?
4. Classification of fixed points in 2-D systems.
5. Sketch a saddle node fixed point.
6. Sketch a stable node fixed point.
7. Sketch an unstable node fixed point.
8. Sketch a stable spiral (fixed point).
9. Sketch an unstable spiral (fixed point).
10. Sketch a center (fixed point).
11. Sketch a stable non-isolated fixed point.
12. Sketch an unstable non-isolated fixed point.
13. What are 2-D homogeneous nonlinear systems?
14. What does it mean that a fixed point is Lyapunov stable?
15. Give an example of Lyapunov stable fixed point.