Lecture 15: Concluding remarks and summary, overview of the course, *exam*

Contents

Demonstrations: Kalliroscope, magnetic pendulum in three magnetic potentials.

1 Comments on your coursework

We start the lecture with a discussion on your coursework evaluation results. The discussion is conducted in an anonymous fashion—no student's results are singled out.

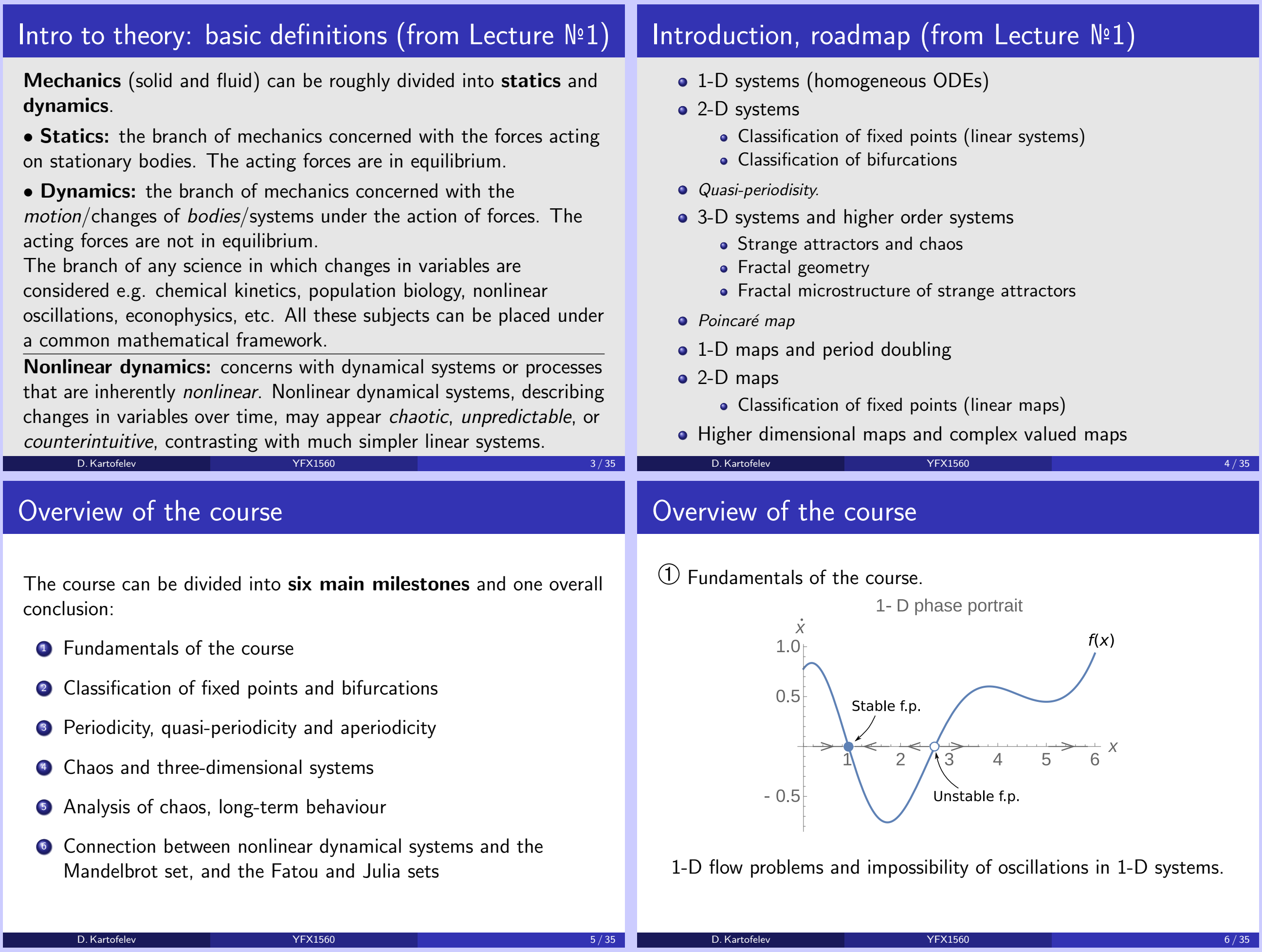
2 Summary and overview of the course

At the end of any achievement it is beneficial to retrospectively look back and reflect on what we have accomplished. The course can be divided into **six main milestones**:

1. Fundamentals of the course
2. Classification of fixed points and bifurcations
3. Periodicity, quasi-periodicity and aperiodicity
4. Chaos and three-dimensional systems
5. Analysis of chaos, long-term behaviour
6. Connection between nonlinear dynamical systems, the Mandelbrot set, and the Fatou and Julia sets

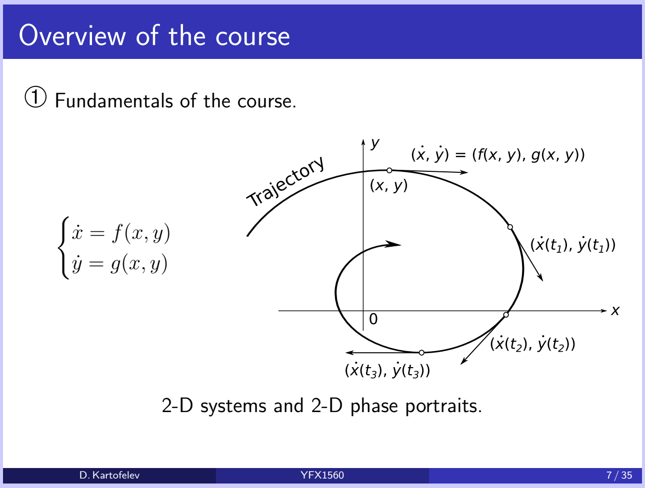
The following slides guide us through the above milestones and **highlights** associated with them:

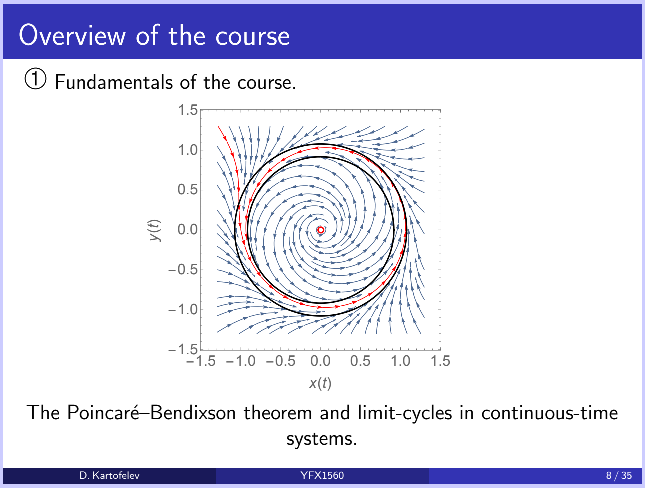
SLIDES: 3–6



Phase portrait plotting proved to be a powerful tool for studying systems without directly integrating them.

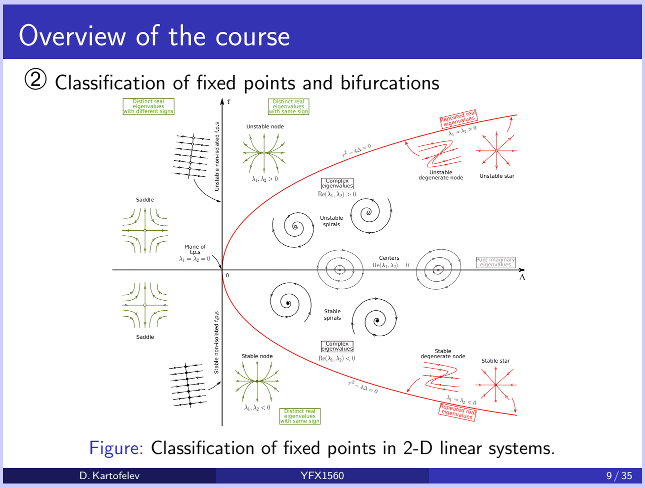
SLIDES: 7–12

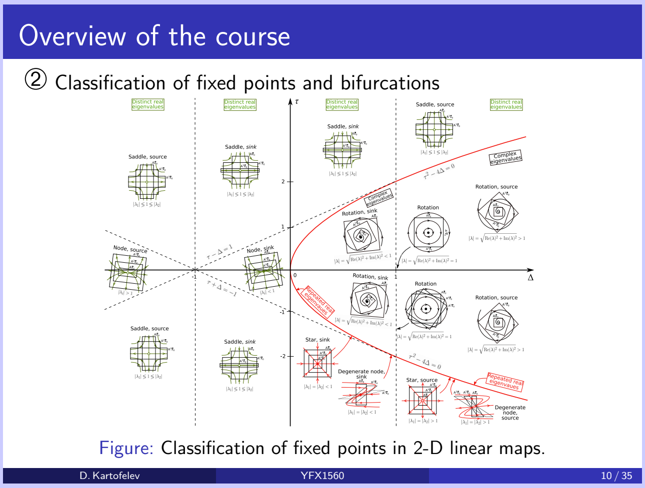


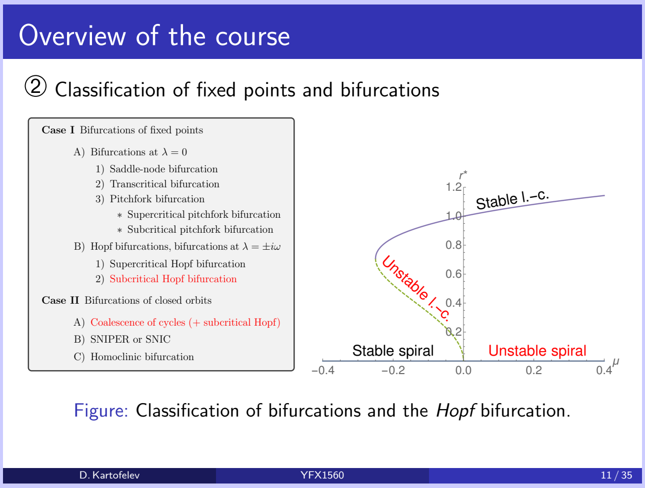


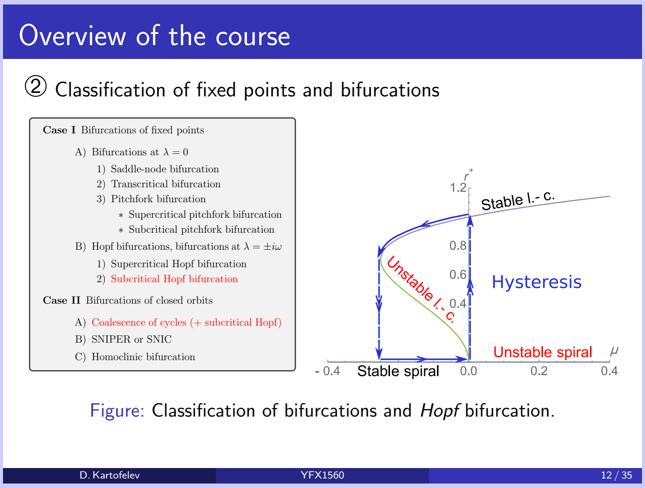
Stable limit-cycles are very important scientifically—they model systems that exhibit self-sustained oscillations.

The Poincaré-Bendixson theorem says that the dynamical possibilities in the phase **plane** are very limited: if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory will approach a closed orbit. Nothing more complicated is possible. In higher-dimensional systems the Poincaré-Bendixson theorem no longer applies. The theorem also implies that **chaos can never occur in the phase plane**—in two-dimensional systems.



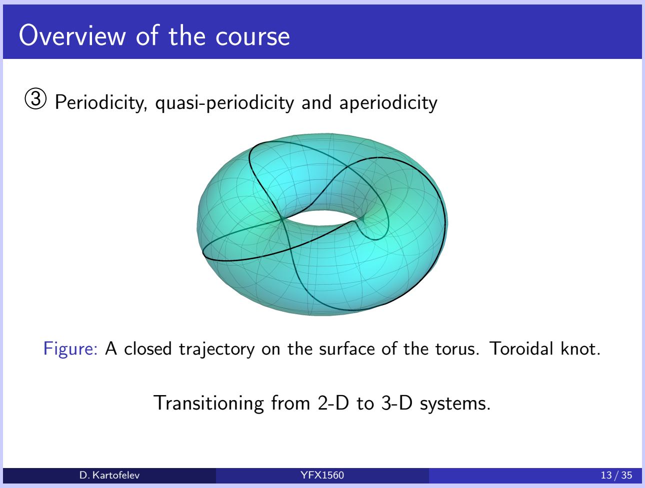


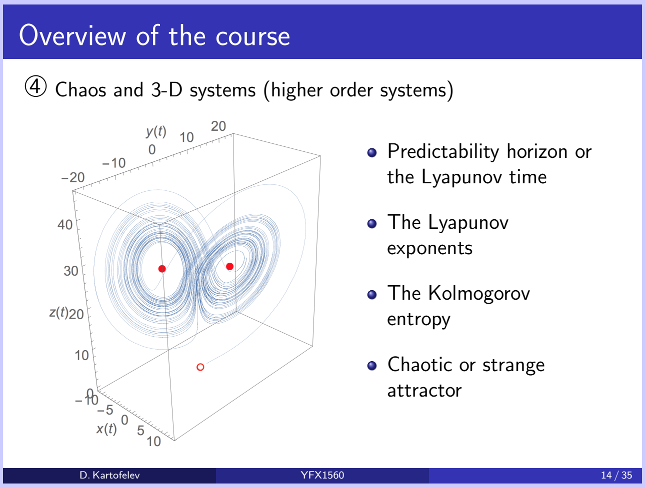




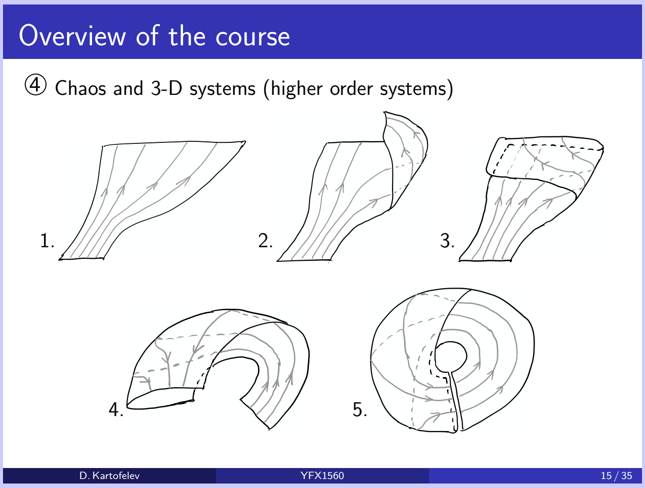
The subcritical Hopf bifurcation can lead to **dramatic failures** of the modelled phenomena. After the bifurcation, the trajectories must jump to a distant attractor, which may be a fixed point, another limit cycle, infinity, or in three and higher dimensions a chaotic attractor.

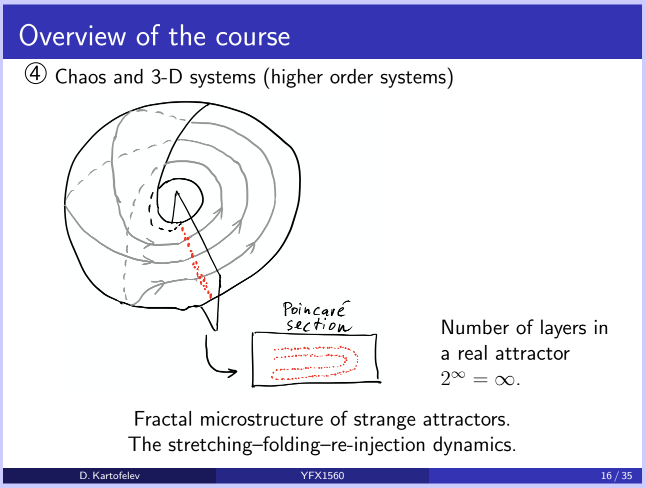
SLIDES: 13–18



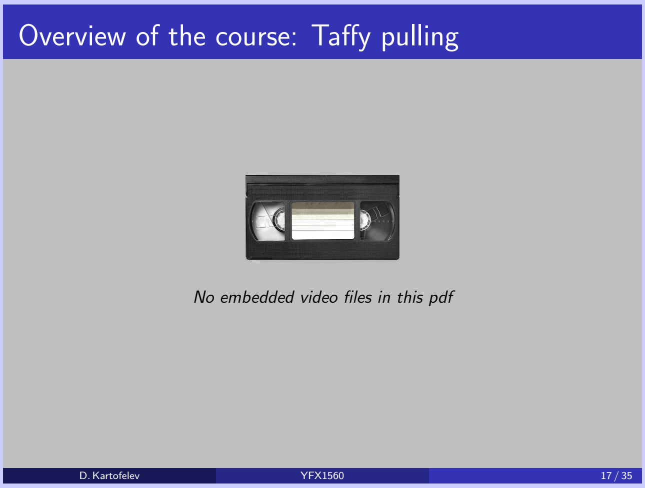


We used quasi-periodicity as a steppingstone between two- and three-dimensional systems. Between steady-state solutions, periodic solutions and (long-term) aperiodic dynamics.



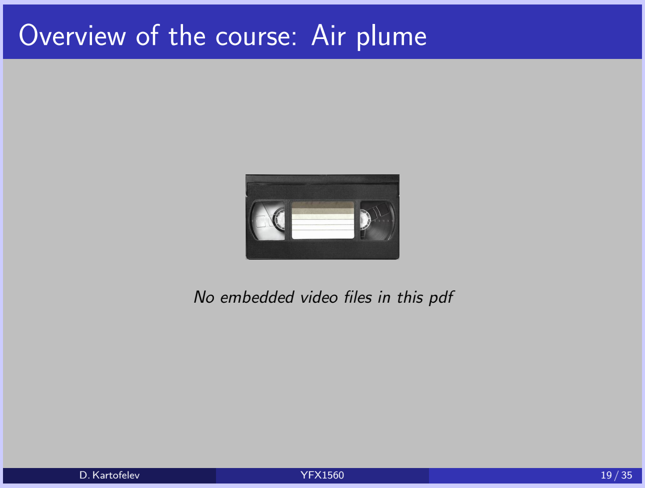


The intuitive understanding of the internal **fractal microstructure of strange attractors** helped us to explain their intricate dynamics.



The **stretching–folding–re-injection dynamics** directly associated with the chaos via the infinite cascade of period doubling bifurcations can be found and directly observed in many natural phenomena. Chaotic dynamics is observable on many scales, from human time and space scale to planetary ones and beyond.

SLIDES: 19–24



Chaos is literally all around us it is never further then a fraction of a millimetre from your skin. Not to mention the chaotic dynamical processes happening inside your body and organs.

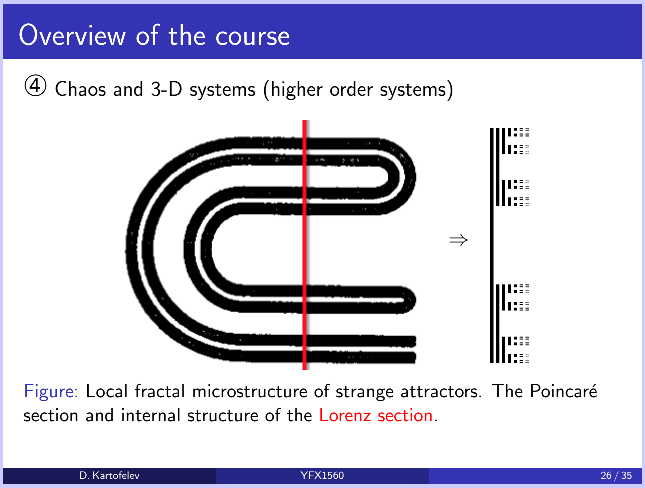
The **stretching–folding–re-injection dynamics** happens on the planetary scale and also on other planets of our Solar system.

Figure: Jupiter. From left to right: the sequence of images taken on Sept. 1, 2017. At the time the images were taken, the spacecraft ranged from 12 143 to 22 908 km from the tops of the clouds of the planet.
Credit: NASA/JPL-Caltech/SwRI/MSSS/Gerald Eichstädt/Seán Doran

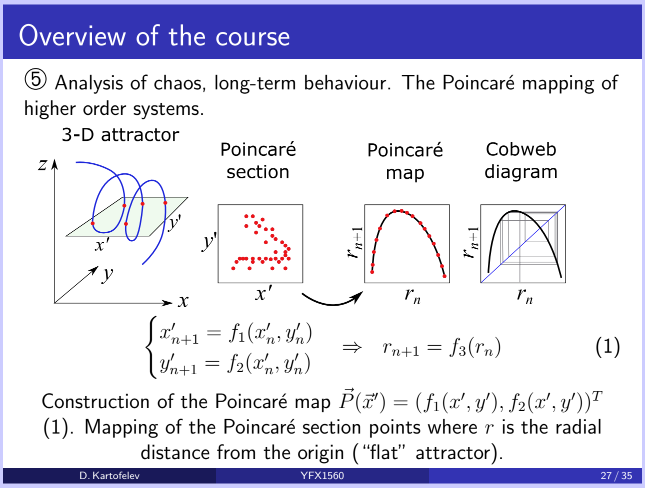
Figure: Left: Image acquired on May 19, 2017, from an altitude of about 33 400 km above Jupiter's cloud tops. Right: Image captured, from an altitude of 18 906 km.
Credit: NASA/JPL-Caltech/SwRI/MSSS/Gerald Eichstädt/Seán Doran

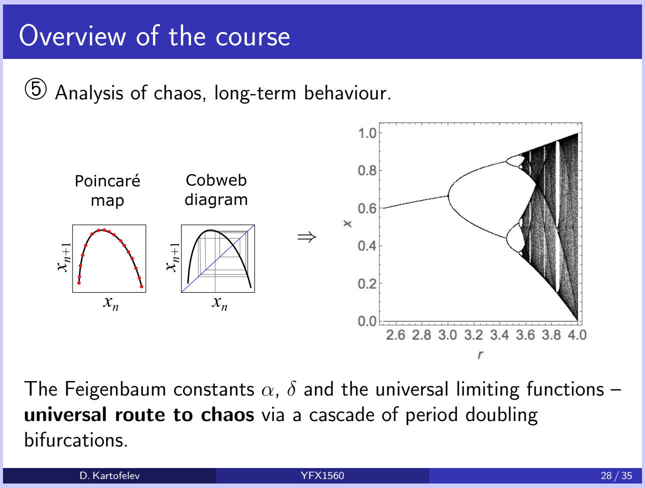
We observed the **stretching–folding–re-injection dynamics** on many scales: from a taffy pulling machine, to a coffee cup, to the air around us, to the weather systems on our planet, and beyond.

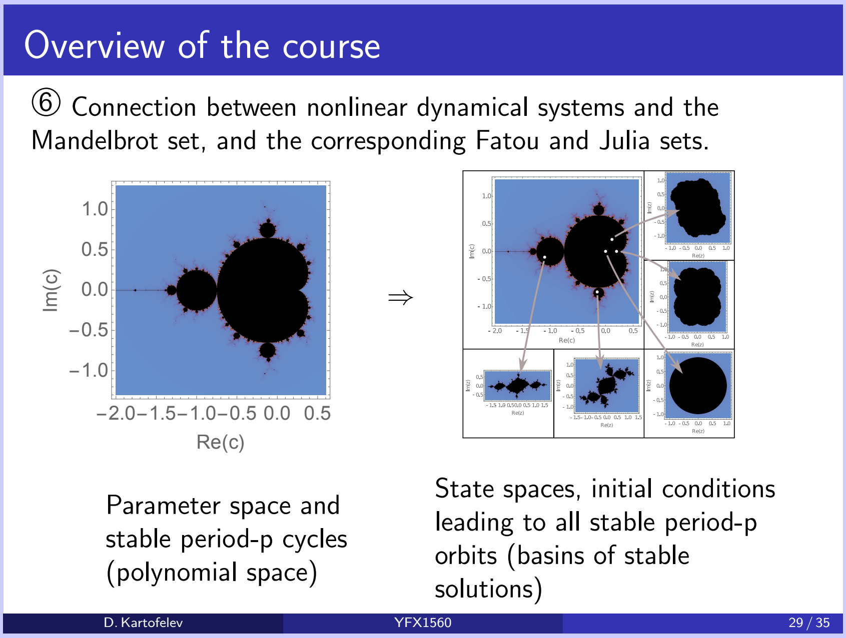
Slides: 25–29



Demonstration: Polar vortices of the aforementioned gaseous giants look similar to the turbulent patterns in the kallisocope toy.







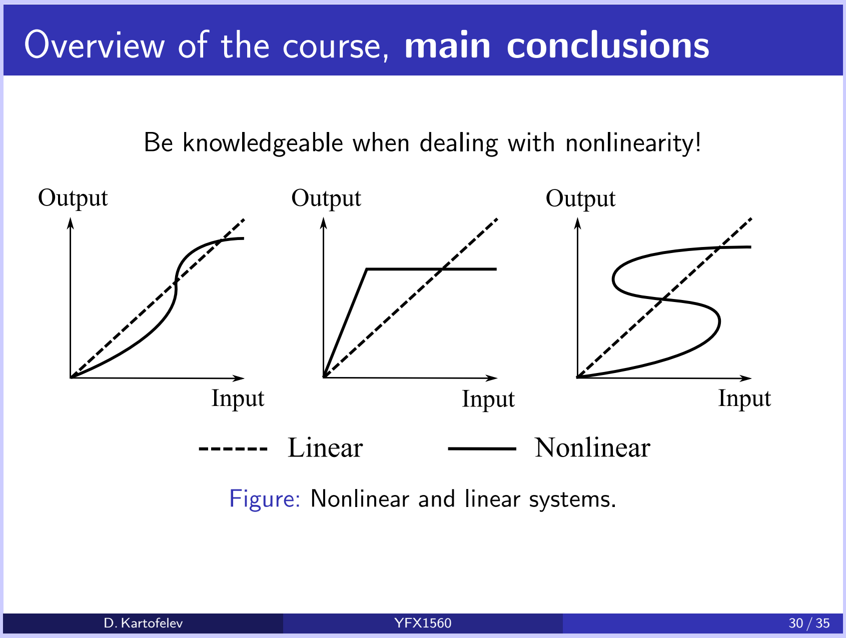
Our study of **discrete-time systems** culminated with the analysis of the complex valued square map $f_c(z, c) = z^2 + c$, where $z, c \in \mathbb{C}$ and the analysis of the Mandelbrot set defined by that map. We showed that nonlinear dynamical systems may exhibit **sensitive dependence on parameters and**

mathematical description itself in addition to the **sensitive dependance on initial conditions** emphasised earlier in the course.

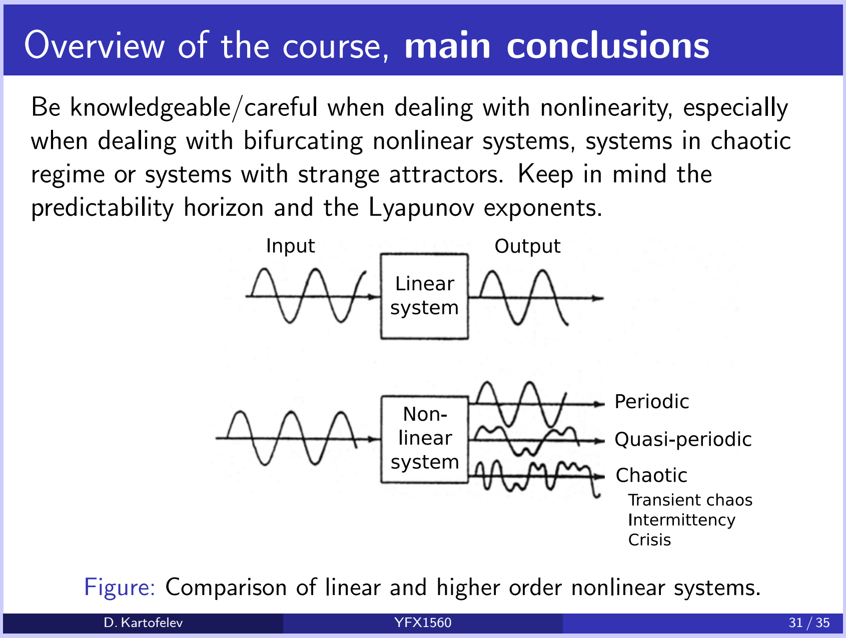
3 Main conclusions of the entire course

We conclude the course with **a single cautionary conclusion**.

SLIDES: 30, 31



Nonlinear systems are different from linear ones. The **dynamics of nonlinear systems can be counterintuitive** and hidden from linear analysis (linearisation). Use **nonlinear analysis methods** presented during this semester to understand nonlinear dynamical systems you encounter in the future.



If there is only one thing you take away from this course, then it should be: **Don't fuck with nonlinearity!**

This concludes our introductory lecture series on nonlinear dynamics and chaos.

4 *Exam, what to expect?*

The following slides give an overview of our final exam that will take place during the exam session:

Slides: 32–34

Exam, what to expect?

A practice mock exam is added to the lecture notes (Lecture 15) and the course webpage.

The final exam consist of:

- **Three** *simple* **questions or definitions** (12 points each) selected from the pool of the revision questions (non verbatim)
- **Three** *simple* **math problems** (for 12 or 13 points each)
- **Two video questions** (13 points each) designed to test your reasoning ability

Maximum points: 100. The exam is graded according to the TalTech grading scale.

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Exam, an example video

Figure: Magnetic pendulum in three magnetic potentials.

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Exam, an example video

No embedded video files in this pdf

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4.1 An example practice exam

1. `[12 p.]` What is chaos in the context of nonlinear dynamics?
2. `[12 p.]` What are the Feigenbaum constants?
3. `[12 p.]` Explain the idea behind construction of a fractal.
4. `[12 p.]` Sketch a 1-D phase portrait corresponding to the following system:

$$\dot{x} = \sin x. \tag{1}$$

5. `[13 p.]` Determine if the following system is linear or nonlinear:

$$\ddot{x} + \dot{x} + \sin x = 0. \tag{2}$$

6. `[13 p.]` Linearise the following system:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = y(x^3 + y^3) + x. \end{cases} \tag{3}$$

7. `[13 p.]` **Video:** Is the magnetic pendulum shown in Fig. 1 and in the video embedded into Slide 32 a chaotic system? If you believe the pendulum to be chaotic, then what is the source or sources of the chaos?
8. `[13 p.]` **Video:** Does the pendulum have an unstable fixed point (unstable pendulum position)? If so, can we position the pendulum at that point for a prolonged period of time? Explain your answers.

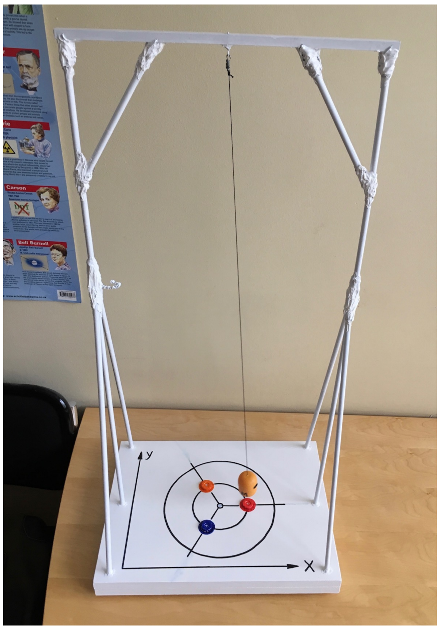
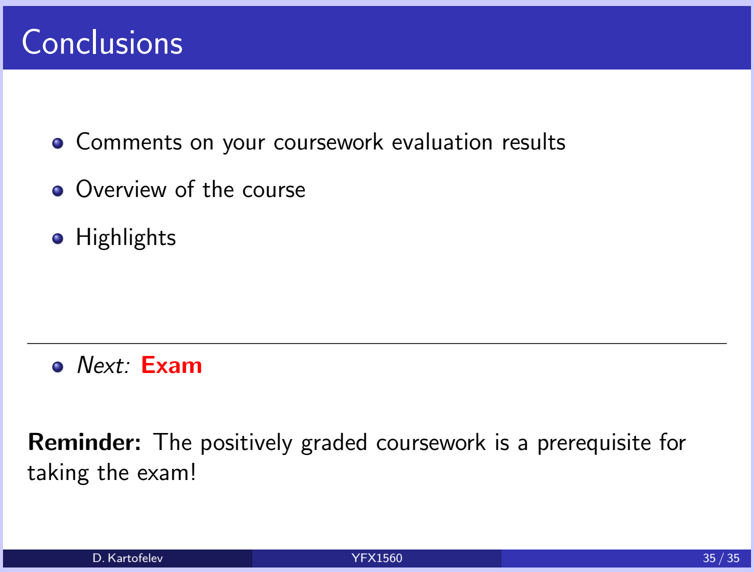


Figure 1: The magnetic pendulum in three magnetic potentials relevant to Questions 7 and 8. The magnets are marked with the red, orange and blue colours.

Demonstration: The magnetic pendulum in three magnetic potentials, shown in Fig. 1, is used to show the dynamics near the unstable fixed point, mentioned in Question 8. A simple modification to the pendulum length can change the stability of that fixed point.

5 Conclusions

Slide: 35



Friendly reminder: Please hand in your courseworks by the deadline—the end of the semester.