Lecture 14: Fractals and fractal geometry, coastline paradox, spectral characteristics of dynamical systems, 1-D complex valued maps, the Mandelbrot set, the Fatou and Julia sets, the Mandelbrot set and nonlinear dynamical systems, introduction to applications of fractal geometry and chaos: synchronisation in nature

Contents

Demonstration: Spontaneous synchronisation of an applauding (clapping) audience.

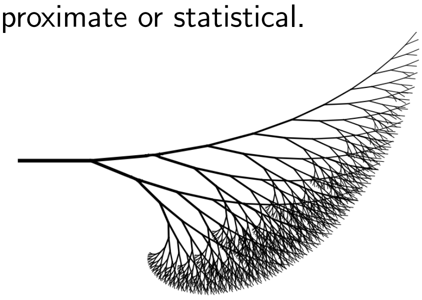
1 Fractals

In Lecture 12 we learned that fractals are subsets of the Euclidean space that can have fractional, i.e., non-integer **similarity dimension** or **box counting dimension**. What does this imply? Let's give fractals a more general definition.

SLIDE: 3

Definition of a fractal[1]

Fractal **Endless** and complex *pattern* with fine structure at arbitrarily small scales. In other words magnification of **tiny features of a fractal are reminiscent of the whole**. Similarity can be exact (invariant), more often it is approximate or statistical.



Examples: The Cantor set, the von Kock curve, the Hilbert curve, the L-systems, etc.

[1]See Mathematica .nb file uploaded to the course webpage.

D. Kartofelev YFX1560 3 / 44

The above slide shows a self-similar L-system iterated to a finite number of iterates, making it not a true fractal according to the given definition. In the literature and in this document such structures will be called **pre-fractals**.

Fractals are not limited to geometric patterns, but can also describe some processes, natural penom-ena, etc. Fractal patterns with various degrees of self-similarity have been rendered or studied in visual, physical, and aural media and found in nature, technology, art, architecture and law. Fractals are of particular relevance in the field of *chaos theory* because they show up in the geometric depictions of most chaotic processes—as **internal structure of strange attractors** or as **boundaries between basins of attraction**.

An **L-system** or the **Lindenmayer system** is a parallel rewriting system and a type of formal grammar. An L-system consists of an alphabet of symbols that can be used to make strings, a collection of production rules that expand each symbol into some larger string of symbols, an initial *axiom* string from which to begin construction, and a mechanism for translating the generated strings into geometric structures. L-systems were introduced and developed in 1968 by Aristid Lindenmayer, a Hungarian theoretical biologist and botanist at the University of Utrecht. Lindenmayer used L-systems to describe the behaviour of plant cells and to model the growth processes of plant development.

The following numerical file was used to generate the pre-fractal shown on Slide 3.

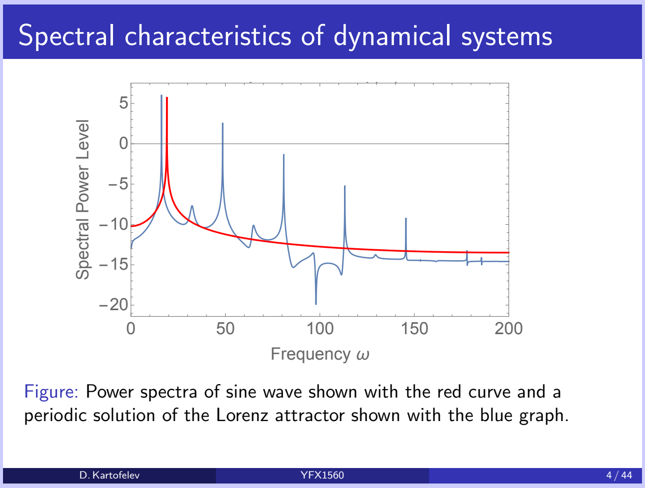
NUMERICS: NB#1

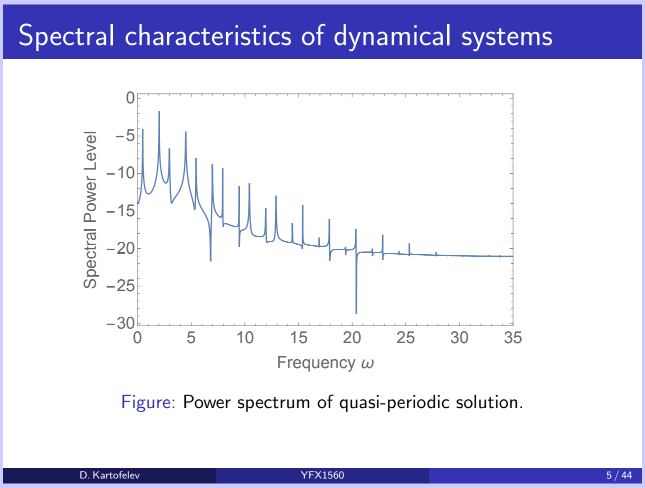
Interactive fractal tree generator.

2 Spectral characteristics of dynamical systems

Below, we provide an overview of the differences in spectral features for different selected types dynamics. All examples shown below have been studied by us in previous lectures.

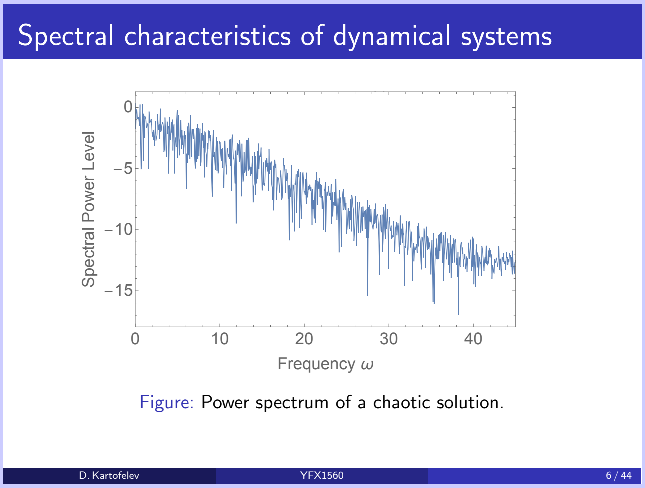
SLIDE: 4–6





The periodic solution of the Lorenz system, originally presented during Lecture 9, used to calculate the spectra shown on Slide 4, uses the following parameter values: $d = 10$, $r = 350$, $b = 8/3$.

The spectra of the quasi-periodic signal, originally presented during Lecture 9, used to calculate the spectra shown on Slide 5, uses the following parameter values: $k_1 = 1.1$, $k_2 = 2.0$, $\omega_1 = 5.02$, $\omega_2 \approx 1.046766$. Here k_1, k_2 are the coupling constants between the two oscillators and ω_1, ω_2 are the natural frequencies of these oscillators.



The strange attractor dynamics of the Lorenz system, originally presented during Lecture 9, used to calculate the spectra shown on Slide 6, uses the following parameter values: $d = 10$, $r = 28$, $b = 8/3$.

Note: Fractal signals feature spectra that are similar to spectra of time-domain signals from chaotic attractors. A fractal signal is by definition similar to itself at all scales, hence the power law—a scaling law.

The spectra shown on Slides 4–5 were calculated using the following numerical file.

NUMERICS: NB#2

Comparison of power spectra of periodic, quasi-periodic, and chaotic signals. The file is featuring the examples used previously throughout the course.

3 1-D complex maps

In this section we are continuing with the idea that maps are simplified models or the Poincaré mappings of three- and higher-dimensional continuous-time systems. The general form of one-dimensional complex valued map is the following:

$$z_{n+1} = f(z_n), \tag{1}$$

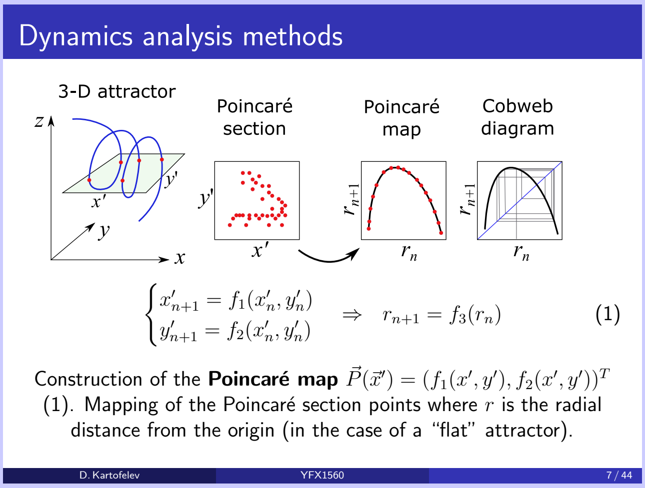
where $f \in \mathbb{C}$ is the given function and $z_n \in \mathbb{C}$. The criteria for finding fixed points z^* and period-p points are the same as were for the real valued maps. One-dimensional complex maps can be represented on the two-dimensional xy-plane, i.e., they can be represented as two-dimensional real maps. Figure 1 show the direct link between the two representations. The real axis is cast onto the horizontal x-axis and the imaginary axis onto the vertical y-axis.

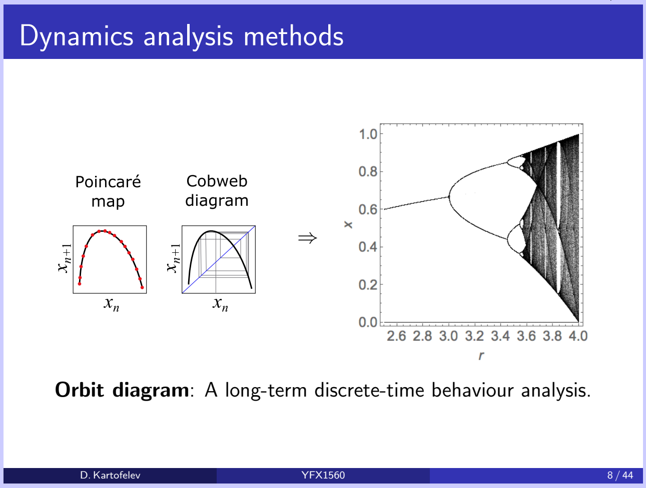
Figure 1: Relationship between a one-dimensional complex map and two-dimensional real map. (Left) Point on the complex plane. (Right) Corresponding equivalent point on the real valued xy-plane where $x_n = \mathrm{Re}\, z_n$ and $y_n = \mathrm{Im}\, z_n$.

3.1 The Mandelbrot set and nonlinear dynamical systems

Let's recap the analysis methods and tools of three- and higher-dimensional systems we have used so far.

Slide: 7, 8





We have used the theory of ordinary differential equations (ODE) to simplify and classify continuous-time systems via linearisation. We have used the numerical calculus (ODE solvers) to analyse continuous-time dynamics. Next, we used the recurrence mapping to study the discrete-time dynamics and long-term stability of chaotic systems by constructing cobweb and orbit diagrams. The above slides gives a schematic overview of dynamics analysis methods used so far.

Curiously enough, heretofore we have studied only real valued solutions. There is no reason to limit ourselves in such a manner. In fact, the general solution of the differential equation is complex. **Also, we haven't payed almost any attention to the control parameters, the other inputs into our systems alongside the initial conditions.** This begs the question, can dynamical systems exhibit sensitive dependants on the control parameter values in addition to the sensitive dependence on initial conditions.

How does a system dynamics depend both on the map function f and control parameters if they exist? In answering this question let's try to think more like mathematicians—be more abstract with our approach. It is obvious that the dynamics of iterated map in the form:

$$z_{n+1} = f_c(z_n, c), \tag{2}$$

where f_c is chosen for simplicity to be a polynomial, $z_n = x_n + iy_n$ is the system state, $c = r + is$ is the free term constant (think: system parameter), and $f_c, z_n, c_n \in \mathbb{C}$, depends on the exact form of polynomial f_c

and initial condition z_0. Let's consider the simplest possible example. We select the lowest order nonlinear polynomial having the free term c—the quadratic polynomial given in the form:

$$f_c(z_n, c) = az_n^2 + bz_n + c, \tag{3}$$

where a, b, $c \in \mathbb{C}$ are the constant coefficients. Further, for $a = 1 + 0\mathrm{i}$, $b = 0 + 0\mathrm{i}$ (we are not interested in the input of linear monomial) we get

$$f_c(z_n, c) = z_n^2 + c, \tag{4}$$

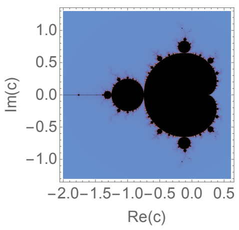
where coefficient c will play the role of the required system parameter we are interested in. This map allows us to study the effects of c on the system dynamics and it also happens to define a well known mathematical object called the **Mandelbrot set**. The following slides give an overview of this complex valued map defined by (4) or more generally by (2).

SLIDES: 9–11

The Mandelbrot set and dynamical systems

The Mandelbrot set[2] M is defined as follows:

$$\begin{cases} z_{n+1} = z_n^2 + c, & \{z, c\} \in \mathbb{C},\ n \in \mathbb{Z}^+ \\ z_0 = 0 \\ c \in M \iff \limsup\limits_{n\to\infty} |z_n| \le 2 \end{cases} \tag{2}$$



[2] See Mathematica .nb file uploaded to the course webpage.

D. Kartofelev YFX1560 9 / 44

The Mandelbrot set

The complex **square map** given in the form:

$$z_{n+1} = z_n^2 + c, \tag{3}$$

where $z = x + \mathrm{i}y$, $c = r + \mathrm{i}s$, and z, $c \in \mathbb{C}$, can be represented as a 2-D real valued map. The component form of (3) is the following:

$$x_{n+1} + \mathrm{i}y_{n+1} = (x_n + \mathrm{i}y_n)^2 + r + \mathrm{i}s, \tag{4}$$

$$x_{n+1} + \mathrm{i}y_{n+1} = x_n^2 + 2\mathrm{i}x_ny_n - y_n^2 + r + \mathrm{i}s, \tag{5}$$

$$x_{n+1} + \mathrm{i}y_{n+1} = x_n^2 - y_n^2 + r + \mathrm{i}(2x_ny_n + s). \tag{6}$$

Separation of the real and imaginary parts, and elimination of the imaginary unit i yields:

$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 + r \\ \cancel{\mathrm{i}}y_{n+1} = \cancel{\mathrm{i}}(2x_ny_n + s) \end{cases} \Rightarrow \begin{cases} x_{n+1} = x_n^2 - y_n^2 + r, \\ y_{n+1} = 2x_ny_n + s, \end{cases} \tag{7}$$

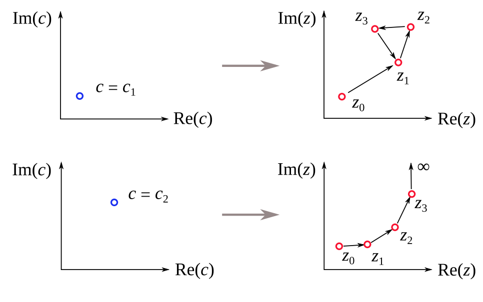
where x, y, r, $s \in \mathbb{R}$.

D. Kartofelev YFX1560 10 / 44

Slide 9 shows a way of visualising the map iterate dynamics as a function of c that is expanded upon below. The black colour corresponds to c values belonging to the Mandelbrot set. The closed circle positioned at the complex plane origin with radius $|c| = 2$ defines the region that can be (not necessarily) mapped by map (4) into itself.

As usual we are interested in the long-term behaviour of iterates z_n, see Slide 11. Two possibilities of the map iterate z_n evolution are possible: The iterates of initial condition z_0 can either settle to a fixed point or period-p point, or escape to infinity (also iterates can stay put in the case of trivial solution).

1-D complex maps, non-trivial dynamics



Fixing the polynomial System dynamics

D. Kartofelev YFX1560 11 / 44

Calculation of the Mandelbrot set itself, shown on Slide 9 or 10, can be summarised in a sentence: The Mandelbrot set is a set of points c in the complex plane (parameter space) for which the iterates z_n of map (4) with initial conditions $z_0 = 0$ do not escape to infinity, see Slide 11. The simplest algorithm for generating the Mandelbrot set is known as the "escape time" algorithm. A repeating map iterate calculation is performed for each c value in the complex plane and based on the behaviour of the iterates, a colour is chosen for that point c (think: image pixel).

Escape time algorithm: The point $c = r + is$ is used to fix the polynomial (4) and the map is iterated for $z_0 = 0$. The values are checked during each iteration n to see whether they have reached the critical "escape" condition $|z_n| > 2$. If that condition is reached, the calculation is stopped, the pixel is drawn, and the next c point is examined. For some c values, escape occurs quickly, after only a small number of iterations. For c values very close to but not in the set, it may take hundreds or thousands of iterations to escape. For values within the Mandelbrot (coloured black) set, escape will never occur (fixed point or period-p point). The programmer must choose how much iteration, or "depth", they wish to examine. The higher the maximal number of iterations, the more detail and subtlety emerge in the final image, but the longer time it will take to calculate the fractal image.

The following file contains interactive code that calculates the set itself and its iterates.

Numerics: nb#3

Complex valued 1-D maps: Generation of the Mandelbrot and corresponding Fatou sets, the Multibrot sets, generation of the Mandelbrot-like complex sets with the corresponding Fatou sets using arbitrary polynomials and rational functions.

Slide: 12

The video shows the process of zooming into the Mandelbrot set and it demonstrates clearly the set's **scale-invariant** self-similar property—**the Mandelbrot set is a *true* fractal**. One can continue zooming into the set for infinite time.

3.2 The Fatou and Julia sets

For each complex polynomial (4), fixed or given by the selection of coefficient c, exists a **basin of all stable period-p cycles**, i.e., a set on initial conditions z_0 (including $z_0 = 0$) that lead to various stable fixed points or period-p points. This set of initial conditions is referred to as **the Fatou set**. Figure 2 shows the Mandelbrot set and the Fatou set corresponding to a selected c value.

In addition to the Fatou sets mathematicians also define **the Julia sets**. Simply put, the Julia set corresponding to fixed c value contains the boundary of the corresponding *nonempty* Fatou set. The Julia set and the Fatou set are two complementary sets. Informally, the Fatou set of the function consists of values with the property that all nearby values behave similarly under repeated iteration of the function (period-p orbits), and the Julia set consists of values such that an arbitrarily small perturbation can cause drastic changes in the sequence of iterated function values. Thus the behaviour of the function on the Fatou set is periodic or stable or regular, while on the Julia set its behaviour is *chaotic* (more on that below).

The following slides gives the definition of the Fatou and the Julia sets for a given polynomial (4) and shows how the Fatou sets look depending on the selection of parameter c.

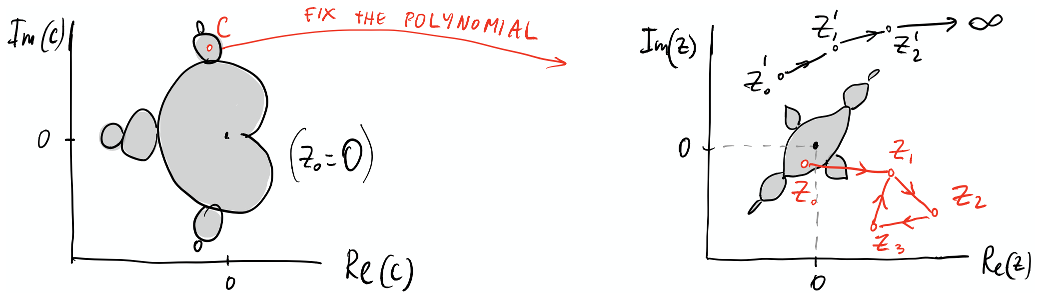


Figure 2: (Left) The Mandelbrot set where the red bullet indicates the selected c value. (Right) The Fatou set corresponding to c shown on the Mandelbrot set on the left. Initial condition z_0 leads to a stable period-3 orbit, in fact all z_0 values in the filled-in or coloured-in region approach some period-3 orbit. Initial condition z_0' escapes to infinity under the iterations and thus is not in the Fatou set.

Slides: 13–16

The Fatou sets and dynamical systems

The Fatou set F_c corresponding to the M set with fixed c value is defined as follows:

$$\begin{cases} z_{n+1} = z_n^2 + c, \quad \{z, c\} \in \mathbb{C}, \; n \in \mathbb{Z}^+ \\ c = \text{const.} = |c| \leq 2 \\ z_0 \in F_c \iff \limsup_{n\to\infty} |z_n| \leq 0.5 + \sqrt{0.25 - |c|} \end{cases} \tag{8}$$

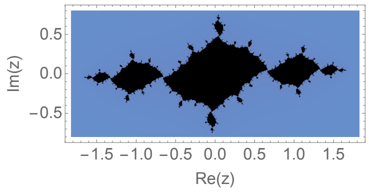


Figure: The Fatou set or the **filled** Julia set for $c = -1.1 - 0.1\mathrm{i}$.

D. Kartofelev YFX1560 13 / 44

The Julia sets

The Julia set J_c corresponding to the M set with fixed c value is defined as follows.

Definition: The Julia set contains the compact boundary of a nonempty Fatou set.

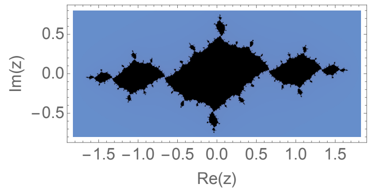
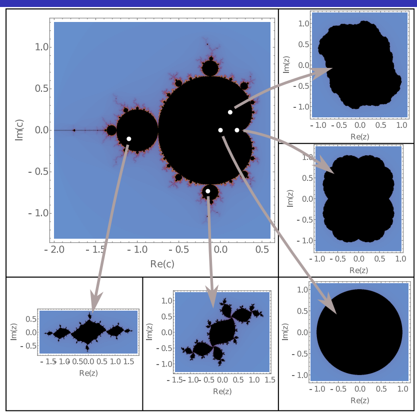


Figure: The Julia set where $c = -1.1 - 0.1\mathrm{i}$. The set is the boundary between the black and blue colours.

D. Kartofelev YFX1560 14 / 44

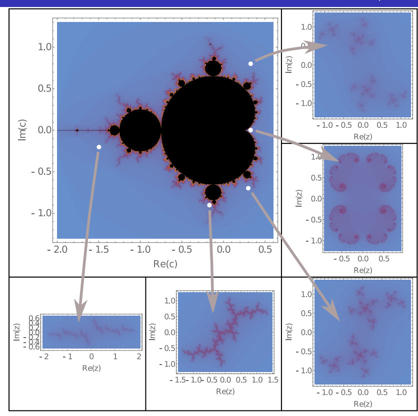
The Julia set is plotted in a very similar manner to the Mandelbrot set except now the c value is fixed and we are analysing initial condition z_0 values within the closed circle of radius $|z| = 0.5 + \sqrt{0.25 - |c|}$ placed at the complex plane origin.

The Mandelbrot set and the Fatou sets



D. Kartofelev YFX1560 15 / 44

The Mandelbrot set and the Fatou sets/Fatou dust



D. Kartofelev YFX1560 16 / 44

The nonempty *filled* Fatou sets correspond to c values that are in the Mandelbrot set, see Slide 15. For c values outside the Mandelbrot set the Fatou sets are not filled and they are referred to as the Fatou dust or *not filled* or *empty* Julia sets, see Slide 16.

The numerical file linked below contains examples of the Fatou sets. The results shown on Slides 13–16 were calculated using this file.

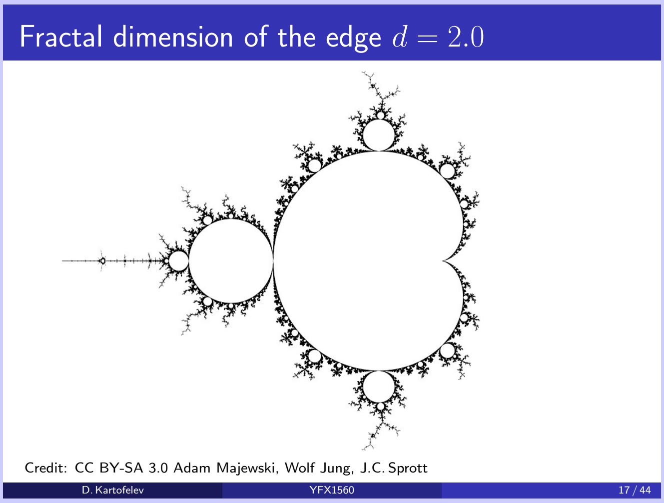
NUMERICS: NB#3

Complex valued 1-D maps: Generation of the Mandelbrot and corresponding Fatou sets, the Multibrot sets, generation of the Mandelbrot-like complex sets with the corresponding Fatou sets using arbitrary polynomials and rational functions.

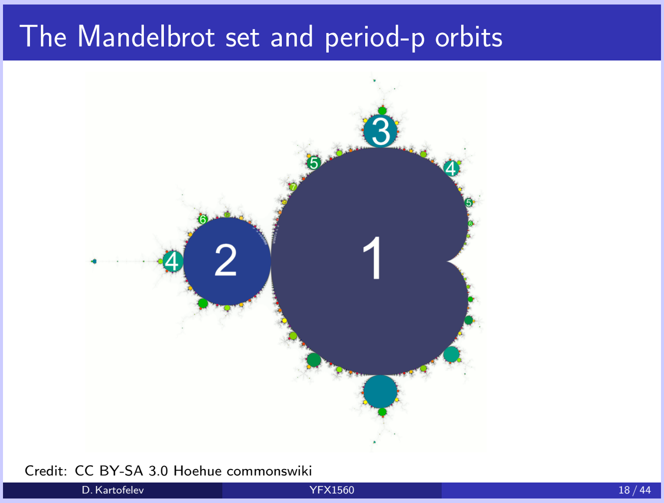
3.3 Dynamics near the edge of the Mandelbrot set and within the Fatou sets

The most *complex* and detailed dynamics of the Mandelbrot and the Fatou sets is found just outside the sets near and on their respective edges, see Slide 17. In the case of the Fatou set this edge is the Julia set. What does the fractal edge mean in terms of the underlying dynamical system? For now, let's focus only on the Mandelbrot set, since we are interested in the effects that the control parameters may have on the resulting dynamics. The conclusions reached below will also hold for the Fatou and Julia sets.

SLIDES: 17, 18



It has been proven that fractal dimension of the edge curve $d = 2.0$. Fractal dimension that happens to be a whole number. The edge of the Mandelbrot fractal is a **plane filling curve** (space filling in higher dimensions). Two simpler examples of space filling curves are presented below.



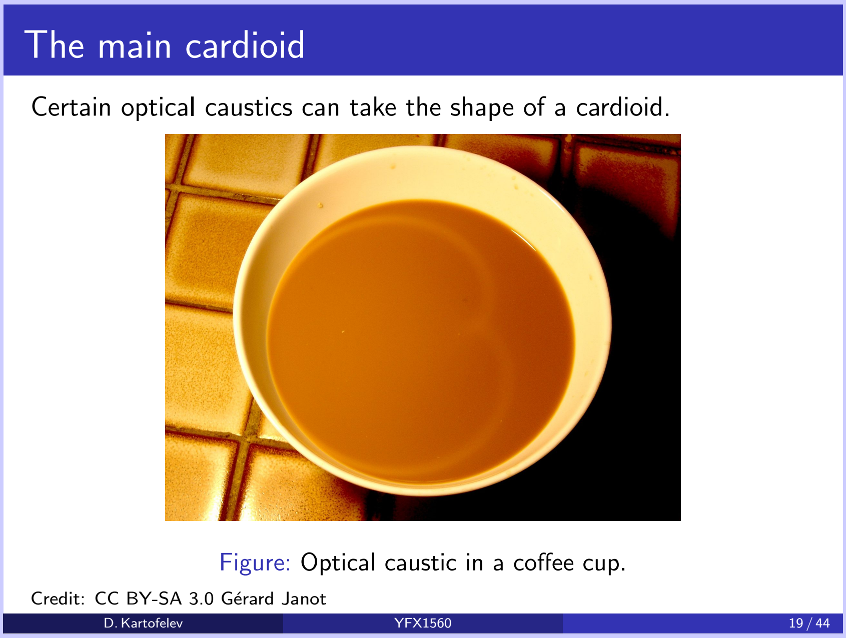
The edge of the Mandelbrot set is populated by so called **period bulbs** corresponding to regions of c populated by stable period-p orbits of iterates z_n with initial condition $z_0 = 0$. Slide 18 shows these regions using numbers, i.e., "2" corresponds to period-2 bulb, etc.

The following numerical file shows two examples of plane filling curves. This interactive visualisation may help one to better visualise and understand the plane filling property of the Mandelbrot set's edge with fractal dimension $d = 2.0$.

Numerics: NB#4

The Hilbert curve: a space filling curve with fractal dimension $d = 2.0$.

Slide: 19



Fun fact: The cardioid at the main bulb of the Mandelbrot set can be found in certain optical caustics.

In order to understand the physical meaning of the fractal edge it is beneficial to study a one-dimensional slice of the set. The simplest one-dimensional subset of the Mandelbrot set is its real axis found for $\operatorname{Im} c = 0$ and $\operatorname{Im} z = 0$. We use the two-dimensional real valued representation of the map function given by (4) that was presented on Slide 10 in the following form:

$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 + r \\ y_{n+1} = 2x_n y_n + s \end{cases} \Rightarrow \begin{bmatrix} y_n = 0 \\ s = 0 \end{bmatrix} \Rightarrow \begin{matrix} x_{n+1} = x_n^2 + r \\ y_{n+1} = 0 \end{matrix} \Rightarrow x_{n+1} = x_n^2 + r, \tag{5}$$

where $x_n = \operatorname{Re} z_n$ and $r = \operatorname{Re} c$. The resulting one-dimensional map can be studied with the aid from the graphical methods we are already familiar with—the cobweb and orbit diagrams.

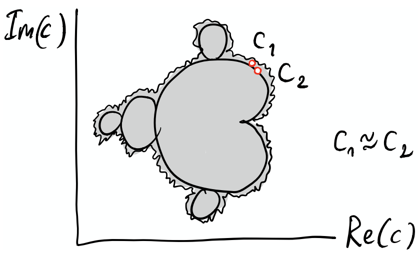


Figure 3: Fractal nature of the edge of the Mandelbrot set is a sign of chaos. The dynamics of two close-by parameter c values, shown with the red points, result in a vastly different iterate z_n dynamics.

The following two numerical files examine the aforementioned slice of the Mandelbrot set (5) using cobweb and orbit diagramming. Careful investigation of the iterates z_n dynamics shows that two close-by c values generate vastly different results. **The fractal nature of the edge is a sign of chaos**, see Fig. 3.

Numerics: NB#5

The Mandelbrot set for $\operatorname{Im}(z) = 0$ and $\operatorname{Im}(c) = 0$: cobweb diagram, orbit diagram and map iterates.

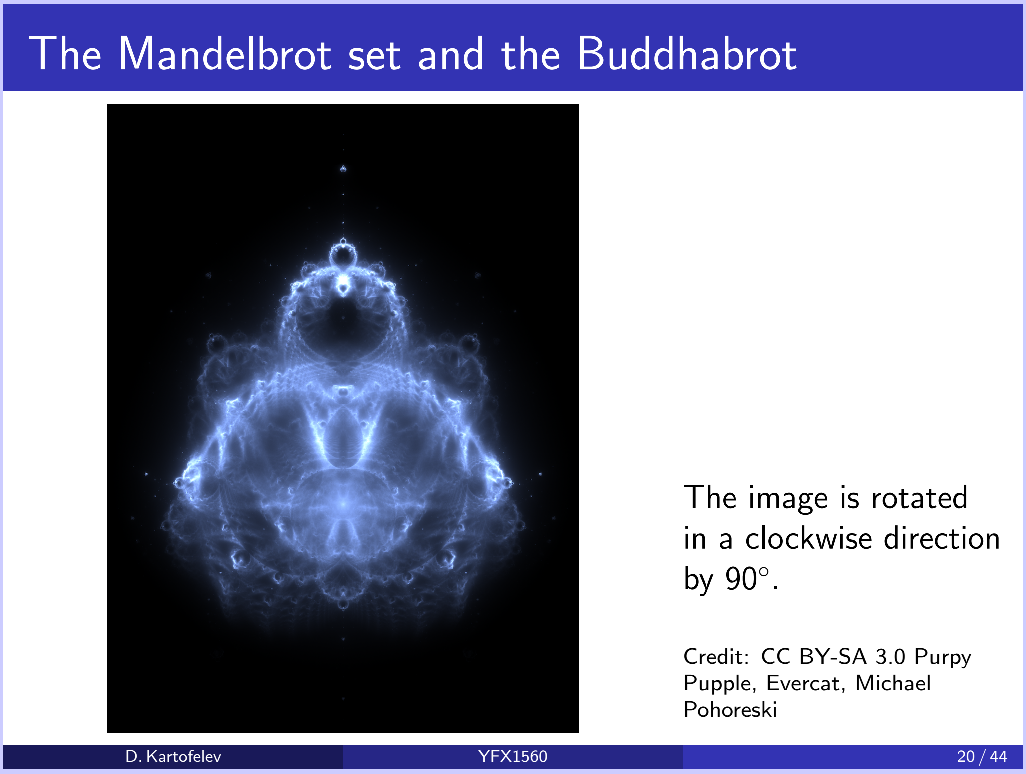
NUMERICS: NB#6
The Mandelbrot set for $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(c) = 0$: cobweb diagram, the set itself and its iterates.

We have shown that nonlinear dynamical systems can exhibit sensitive dependence on parameter values (think: physical description, physical constants, external forcing, etc.) that leads to a **chaotic dynamics in the underlying continuous-time systems** not only in the Mandelbrot and other similar sets.

As stated above the reached conclusions also hold for the Fatou and Julia sets. In many instances the Fatou set may also have a fractal edge. This indicates that the initial values z_0 located there leads to chaotic solutions (even for c values located in the Mandelbrot set) and exhibit sensitive dependence on initial conditions—the fact previously known to us.

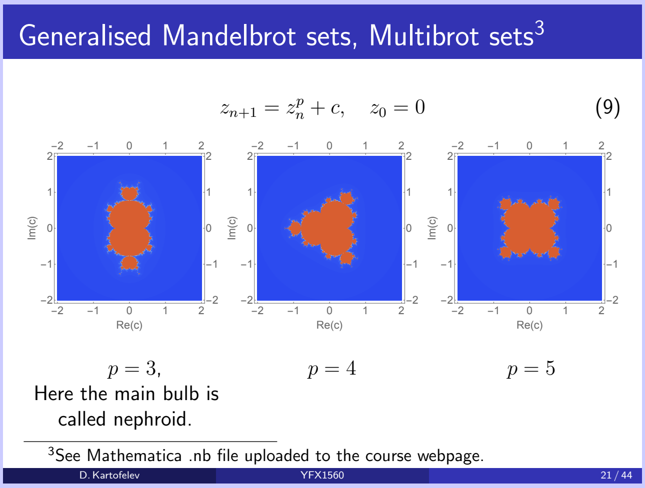
3.4 Buddhabrot set and the generalised Mandelbrot sets

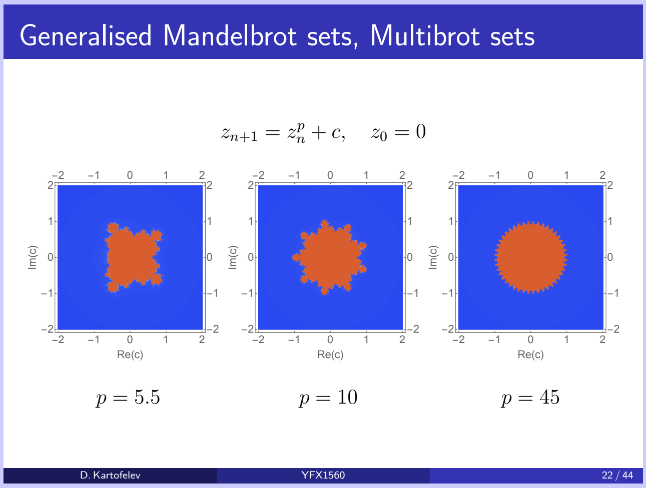
SLIDES: 20–22



Buddhabrot is a fractal rendering technique related to the Mandelbrot set. The name reflects its pareidolic resemblance to classical depictions of Gautama Buddha, seated in a meditation pose with a forehead mark *tikka* and traditional topknot *ushnisha*.

Image rendering algorithm: The Buddhabrot image can be constructed by first creating a two-dimensional array of boxes, each corresponding to a final pixel in the image. Each box (i, j) for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ has size in complex coordinates of Δx and Δy, where $\Delta x = w/m$ and $\Delta y = h/n$ for an image of width w and height h. For each box, a corresponding counter is initialised to zero. Next, a random sampling of c points are iterated through the Mandelbrot map. For points which *do* escape within a chosen maximum number of iterations, and therefore are *not* in the Mandelbrot set, the counter for each box entered during the escape to infinity is incremented by 1. In other words, for each sequence corresponding to c that escapes, for each point z_n during the escape, the box that z_n lies within is incremented by 1. Points which do not escape within the maximum number of iterations (and considered to be in the Mandelbrot set) are discarded. After a large number of c values have been iterated, grayscale shades are then chosen based on the distribution of values recorded in the array. The result is a density plot highlighting regions where z_n values spend the most time on their way to infinity.





The generalised Mandelbrot or **multibrot sets** are shown above. For power $p \to \infty$ the set becomes progressively more circular.

One can generate Mandelbrot-like sets for arbitrary rational functions, and find the Fatou and Julia sets that correspond to these mapping functions. The numerical file linked below contain a code that can generate such sets.

Numerics: nb#3

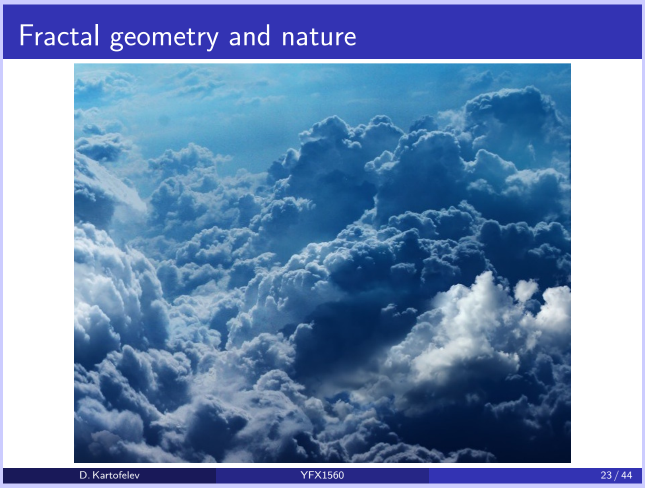
Complex valued 1-D maps: Generation of the Mandelbrot and corresponding Fatou sets, the Multibrot sets, generation of the Mandelbrot-like complex sets with the corresponding Fatou sets using arbitrary polynomials and rational functions.

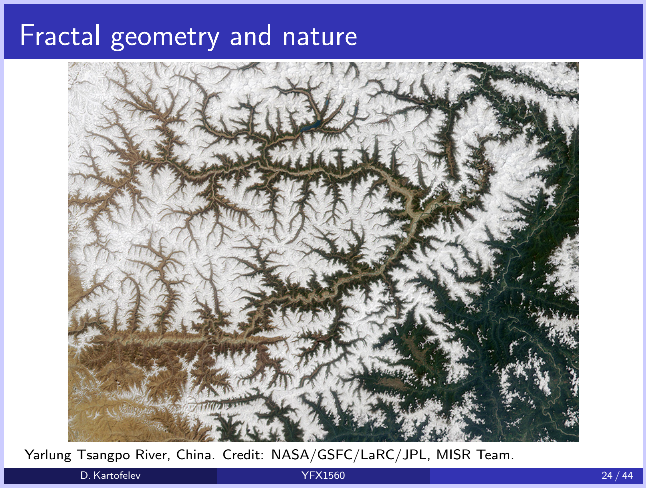
4 Fractal geometry and chaos in nature and applications

4.1 Pre-fractal structures in nature

Clouds are not spheres, mountains are not cones, and lightning does not travel in a straight line. Following slides show some examples of fractal geometry present in nature.

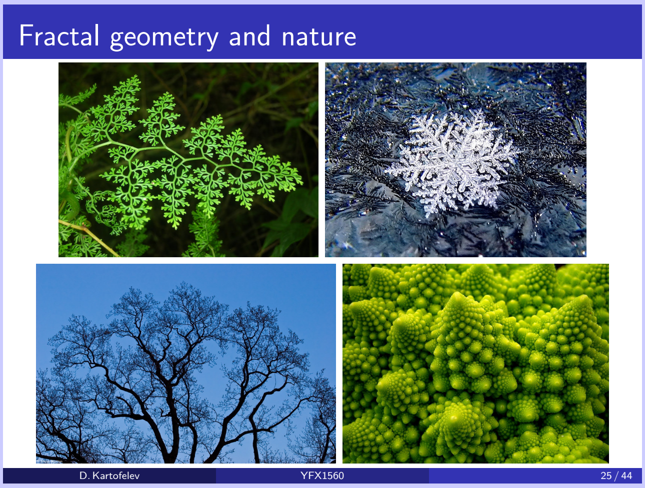
Slides: 23–27

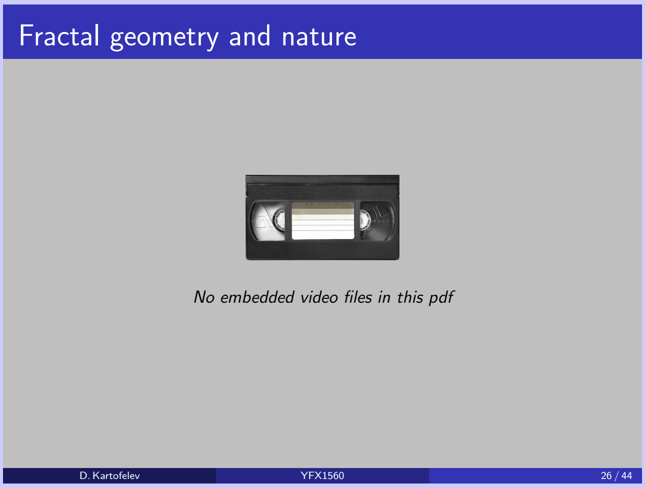


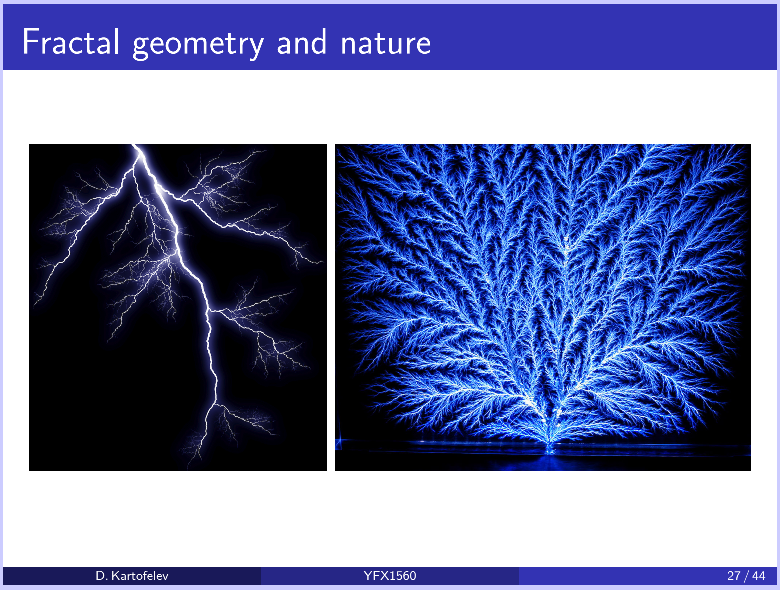


The climate, weather and the cloud formation, shown above, are all chaotic processes. We say that natural phenomena shown here and below have **pre-fractal geometry** because these natural phenomena are not self-similar at infinitely many scales, there exists only a finite range of scales where the fractal similarity dimension d holds, see Slide 3.

Note: Fractal geometry is not the geometry of nature if we define it strictly as an infinitely scalable and self-similar pattern.

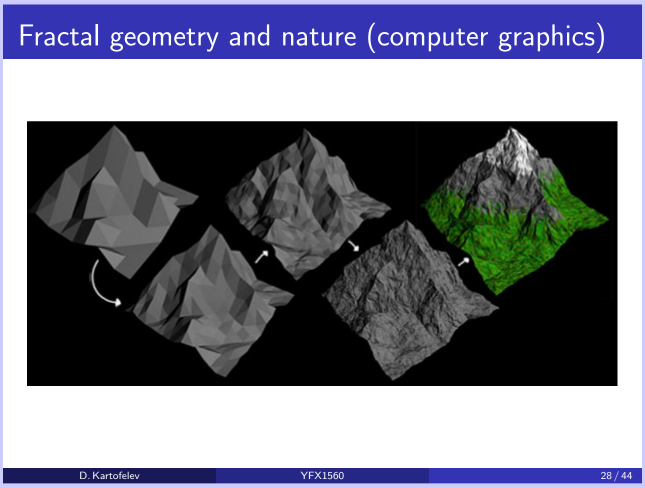


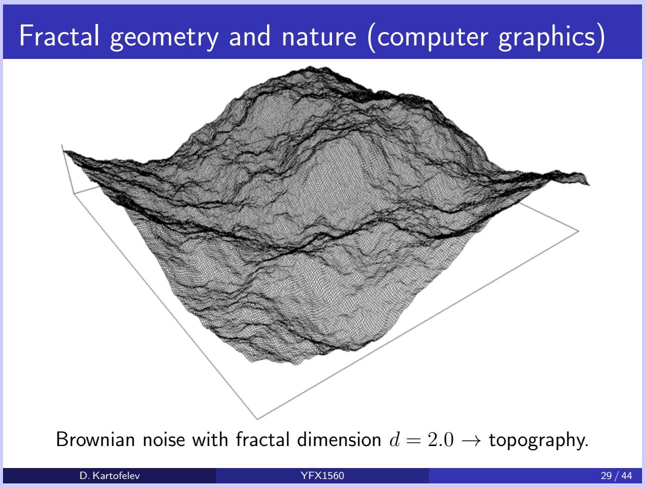




4.2 Applications of fractal geometry and *chaos theory*

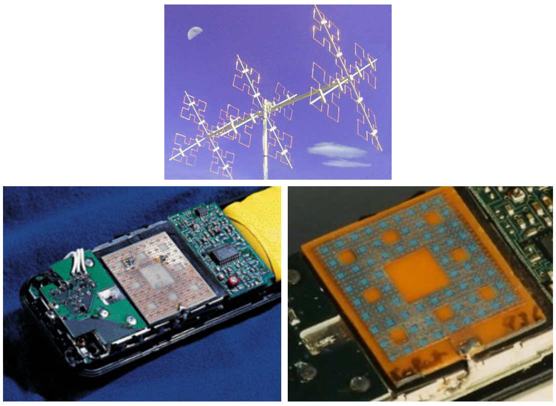
Slides: 28–30





Here again, the generated topography are pre-fractals and that is the reason why they are similar to natural landscapes that do not exhibit self-similarity at infinitely many scales.

Fractal geometry and technology



D. Kartofelev YFX1560 30 / 44

A **fractal antenna** is an antenna that uses a self-similar design to maximise the effective length, or increase the perimeter (on inside sections or the outer structure), of material that can receive or transmit electromagnetic radiation within a given total surface area or volume.

In addition to the above examples fractal generation algorithms are widely used in various application:

- Modelling of natural structures
- Botanical plant structures
- Image compression in computer graphics
- Analysis of medical diagnostic images
- Applications in engineering and architecture
- Study of convergence of iterative processes and of chaotic phenomena
- Fractal art, including music

Some applications of the chaos theory include:

- Cryptography (image encryption algorithms, hash functions, secure pseudo-random number generators, stream ciphers, watermarking and steganography)
- Robotics (passive walking biped robots)
- Biology (model of intrauterine hypoxia, discrete population dynamics)

4.2.1 Coastline paradox

Fractal geometry helps to understand a potentially confusing natural phenomena.

SLIDES: 31–37

Coastline paradox

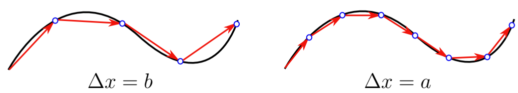
The **coastline paradox**[4] is the counterintuitive observation that the coastline of a landmass does not have a well-defined length. This results from the **fractal-like properties of coastlines**. The first recorded observation of this phenomenon was by Lewis Fry Richardson and it was expanded by Benoit Mandelbrot.

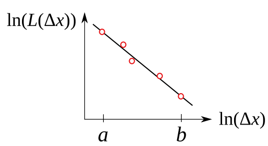
Read: B. Mandelbrot, "How long is the coast of Britain? Statistical self-similarity and fractional dimension," *Science, New Series*, **156**(3775), 1967, pp. 636–638.

[4]See Mathematica .nb file uploaded to the course webpage.

D. Kartofelev YFX1560 31 / 44

Coastline paradox





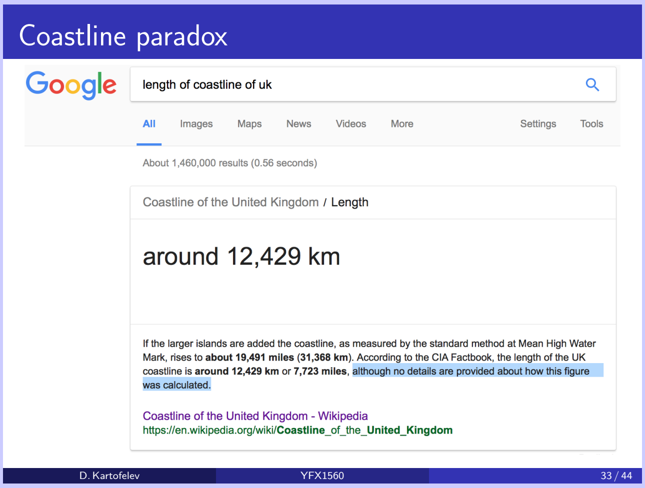
Slope of the resulting graph:

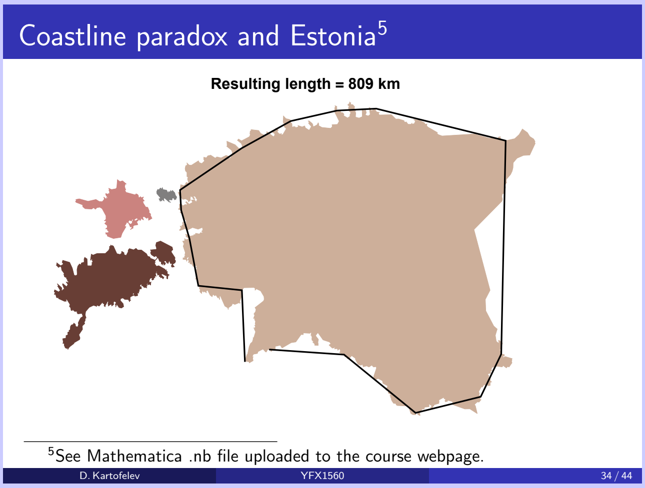
$$d = -\frac{\ln(L(\Delta x))}{\ln(\Delta x)} = \frac{\ln(L(\Delta x))}{\ln(1/\Delta x)}, \quad (10)$$

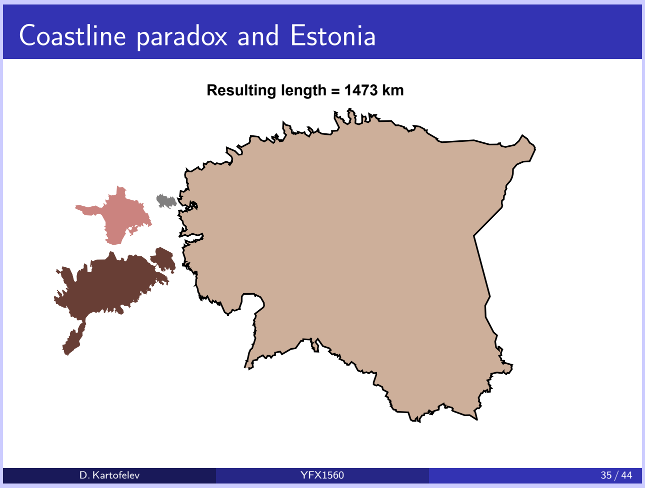
where L is the resulting measurement and Δx is the measurement resolution, i.e., length of a measuring stick.

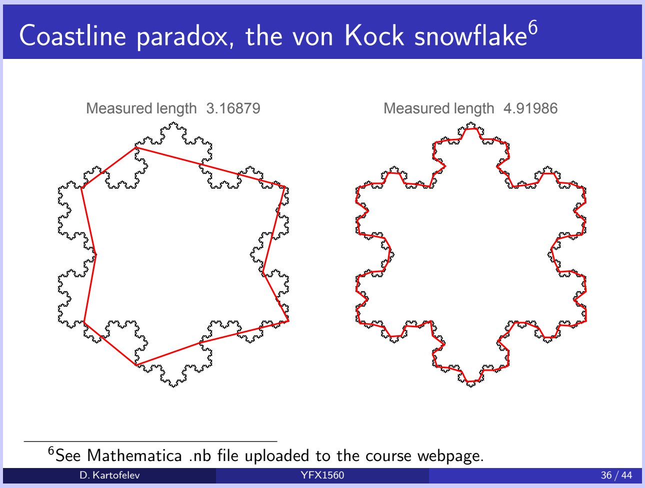
D. Kartofelev YFX1560 32 / 44

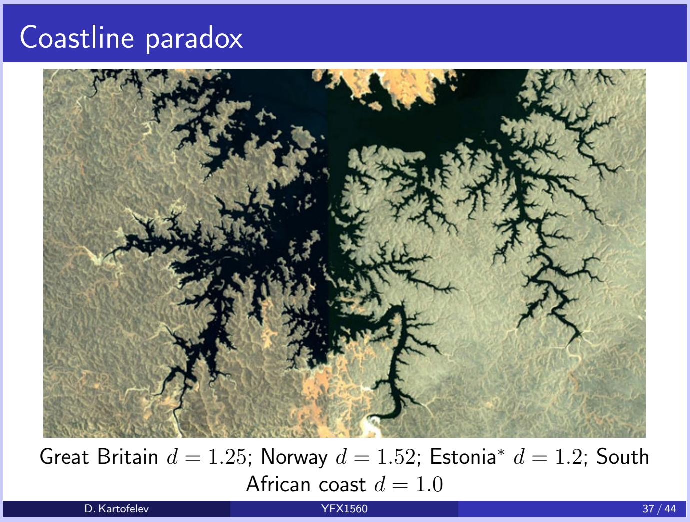
The differently sized "measuring sticks" result in different costal lengths. The same principle applies to rivers and other pre-fractal natural phenomena.











The following interactive numerical file demonstrates how the circumference of the von Koch snowflake (with finite number of iterations) depends on the measurement resolution, see Slide 36.

Numerics: nb#7

Interactive program that plots and measures the length of the von Koch snowflake pre-fractal using different measurement resolutions.

The following interactive numerical file demonstrates how the circumference of the Estonian mainland coastline depends on the measurement resolution, see Slides 34 and 35.

Numerics: nb#8

Interactive program that plots and measures the length of the Estonian coastline using different measurement resolutions.

Reading suggestion

Link	File name	Citation
Paper#1	paper7.pdf	Benoit Mandelbrot, "How long is the coast of Britain? Statistical self-similarity and fractional dimension," *Science, New Series*, **156**(3775), pp. 636–638, (1967). Stable URL: www.jstor.org/stable/1721427

4.2.2 Synchronisation

The analysis methods of nonlinear dynamics and chaos theory shed light on the seemingly spontaneous **synchronisation** phenomena. The following slides show examples of synchronisation.

Slides: 38–41

Synchronisation: metronomes

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D. Kartofelev YFX1560 38 / 44

Synchronisation: fireflies

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D. Kartofelev YFX1560 39 / 44

Synchronisation: The Millennium bridge (2000)

No embedded video files in this pdf

D. Kartofelev YFX1560 40 / 44

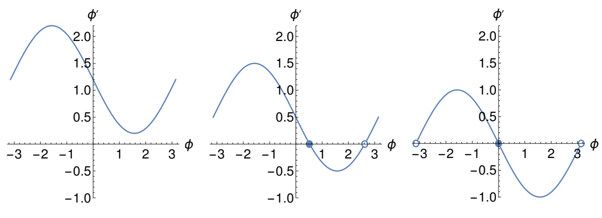
Synchronisation

Aperiodicity of chaos → bifurcation/s → periodic solution

Conceptual model:

$$\dot{\phi} = \mu - \sin\phi, \tag{11}$$

where $\mu \geq 0$ is the system parameter and ϕ is the phase difference/s.



$\mu > 1$ $\quad 0 < \mu < 1$ $\quad \mu = 0$

D. Kartofelev YFX1560 41 / 44

By using the bifurcation theory we can figure out how to restore or reverse chaotic dynamics to order (stable fixed point, period-p dynamics or periodic continuous-time dynamics). The synchronisation is usually driven by feedback or coupling present in a system.

Demonstration: A group of people in a room are capable of synchronising their clapping in a rather short time. Try it. More surprisingly, applauding audiences can synchronise spontaneously without planning it in advance.

Revision questions

1. Define fractal (technical definition).
2. Define pre-fractal.
3. Explain the coastline paradox.
4. Can a coastline be described with Euclidean geometry?
5. What determines spectral characteristics of dynamical systems?
6. What is a 1-D complex valued map?
7. What are the Mandelbrot set and the Fatou sets?
8. What is the Julia set?
9. Assuming $z = x + \mathrm{i}y$, $c = r + \mathrm{i}s$, and $z,\, c \in \mathbb{C}$, show that map in the form

$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 + r, \\ y_{n+1} = 2x_n y_n + s, \end{cases} \tag{6}$$

is the real counterpart of the Mandelbrot set.

10. What is the physical meaning of the Mandelbrot set?
11. What is the physical meaning of the Fatou sets?
12. What is the generalised Mandelbrot set also known as the Multibrot set?
13. Name an example of self-similar phenomena in nature.