Lecture 10: 1-D maps, the Lorenz map, the logistic map, sine map, period doubling bifurcation, tangent bifurcation, transient and intermittent chaos in maps, orbit diagram (or the Feigenbaum diagram), the Feigenbaum constants, universals of unimodal maps, universal route to chaos

Contents

Handout: Orbit diagram of the logistic map.

1 The Lorenz map

In this lecture we continue to investigate **the possibility that the Lorenz attractor might be long-term periodic**. As in previous lecture, we use the one-dimensional Lorenz map in the form:

$$z_{n+1} = f(z_n), \tag{1}$$

where f was *incorrectly* assumed to be a continuous function, to gain insight into the continuous time three-dimensional flow of the Lorenz attractor given by

$$\begin{cases} \dot{x} = d(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{2}$$

where parameters $r = 28$, $d = 10$, and $b = 8/3$ are the control parameters that result in the system being in a chaotic regime—more accurately, these parameters lead to the strange attractor introduced last week.

1.1 Cobweb diagram and map iterates

The **cobweb diagram**, also called the **cobweb plot**, introduced in the previous lecture is a graphical way of thinking about and determining the iterates of maps including the Lorenz map, see Fig. 1. In order to construct a cobweb plot move **vertically to function $f(z)$ and register the function value, move horizontally to the diagonal and repeat**, see Lecture 9 for basic examples.

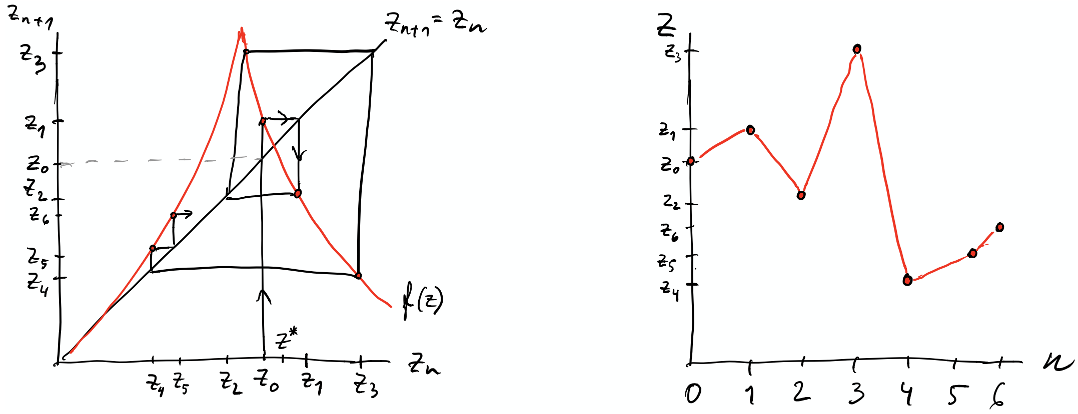

Figure 1: (Left) Cobweb diagram of the Lorenz map. (Right) The Lorenz map iterates corresponding to the cobweb diagram shown on the left.

In the previous lecture we showed, using the linearisation procedure, that the **fixed point z^*** (period-1 point) satisfying

$$\boxed{f(z^*) = z^*,} \tag{3}$$

where $f(z)$ is the function defining the Lorenz map, is unstable. The general conditions for fixed point stability were shown to be dependant on the absolute value of map function slope evaluated at given fixed point z^*. For

$$|f'(z^*)| < 1, \tag{4}$$

the fixed point is stable. For

$$|f'(z^*)| = 1, \tag{5}$$

the fixed point is participating in a bifurcation. An example of this is shown below in Sec. 3.2. For

$$|f'(z^*)| > 1, \tag{6}$$

the fixed point is unstable.

The following interactive numerical file shows the Lorenz map, its cobweb diagram and iterates. It can be used to confirm the fact that the Lorenz map's fixed point z^* is indeed unstable.

NUMERICS: NB#1

Cobweb diagram and iterates of the (generalised) Lorenz map. Lyapunov exponent $\lambda(r)$ of the logistic map.

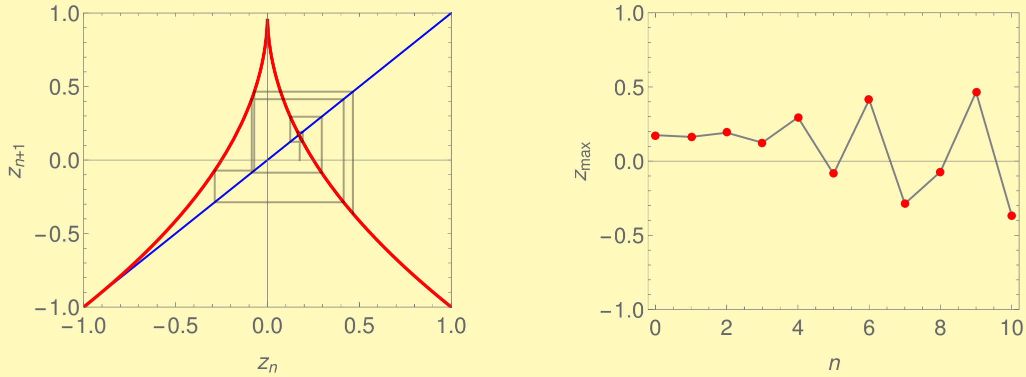

Figure 2: (Left) Cobweb diagram of the normalised Lorenz map showing the first ten iterates for $z_0 \approx z^*$ where the fixed point z^* is unstable, since $|f'(z^*)| > 1$. The Lorenz map has property $|f'(z)| > 1$, $\forall z$ in the map basin. (Right) Lorenz map iterates corresponding to the cobweb diagram shown on the left.

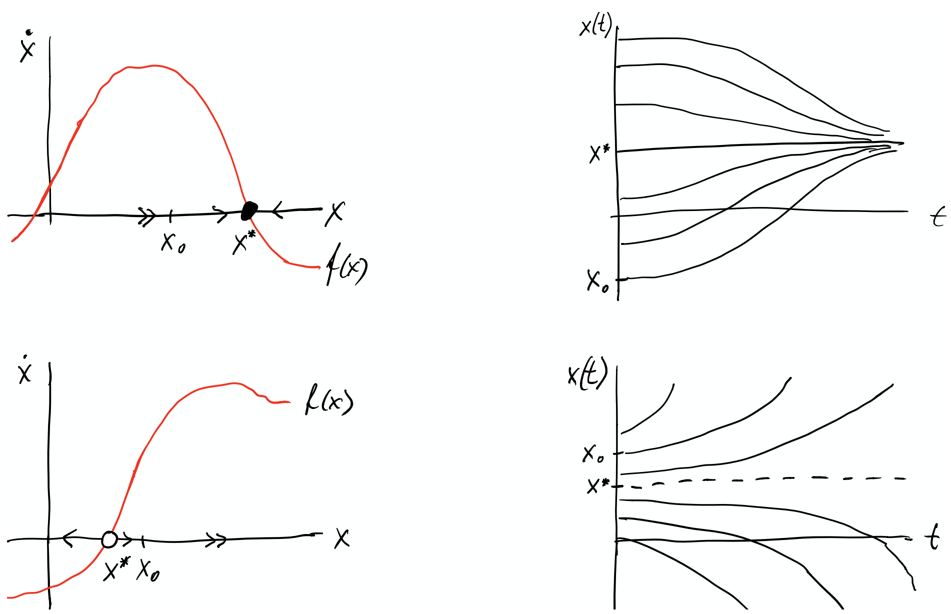

Figure 3: (Top-left) Stable fixed point x^* of a one-dimensional system given by $\dot{x} = f(x)$ where $f'(x^*) < 0$ is shown with the filled bullet. (Top-right) A family of time-series solutions shown for several initial conditions positioned close to x^*. The solutions correspond to the phase portrait shown on the left. (Bottom-left) Unstable fixed point x^* shown with the empty bullet where $f'(x^*) > 0$. (Bottom-right) A family of time-series solutions shown for several initial conditions positioned close to x^*. The solutions correspond to the phase portrait shown on the left.

1.2 Comparison of fixed point x^* in 1-D continuous-time systems and 1-D discrete-time maps

The fixed points of one-dimensional continuous-time systems or in other words ordinary differential equations (ODEs) and of discrete-time one-dimensional maps are clearly analogous. Below we'll start to distinguish fixed points of maps by their type. The type of the fixed point mentioned here will be referred to as so called period-1 point. Figures 3 and 4 show that analogy and other similarities.

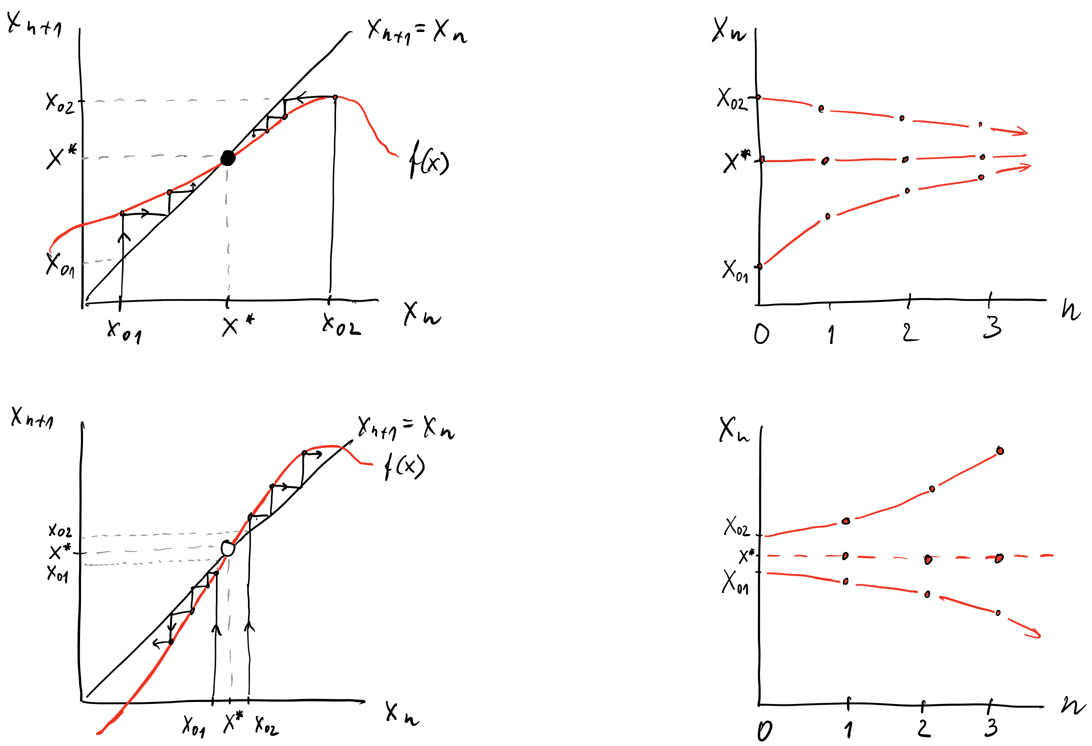

Figure 4: (Top-left) Stable fixed point x^* (a period-1 point) of a one-dimensional map given by $x_{n+1} = f(x_n)$ where $|f'(x^*)| < 1$ is shown with the filled bullet. (Top-right) Three sets of map iterates x_n where $n \in [0, 3]$ shown for initial conditions x_{01}, x^* and x_{02} corresponding to the cobweb diagram shown on the left. (Bottom-left) Unstable fixed point x^* (a period-1 point) shown with the empty bullet where $|f'(x^*)| > 1$. (Bottom-right) Three sets of map iterates x_n shown for initial conditions x_{01}, x^* and x_{02} corresponding to the cobweb diagram shown on the left.

1.3 Period-p orbit and stability of period-p points

We ended the previous lecture with two open ended questions: Can trajectories in a cobweb diagram of the Lorenz map close onto themselves? And, how does a closed trajectory in the cobweb plot translate into the three-dimensional continuous-time Lorenz flow?

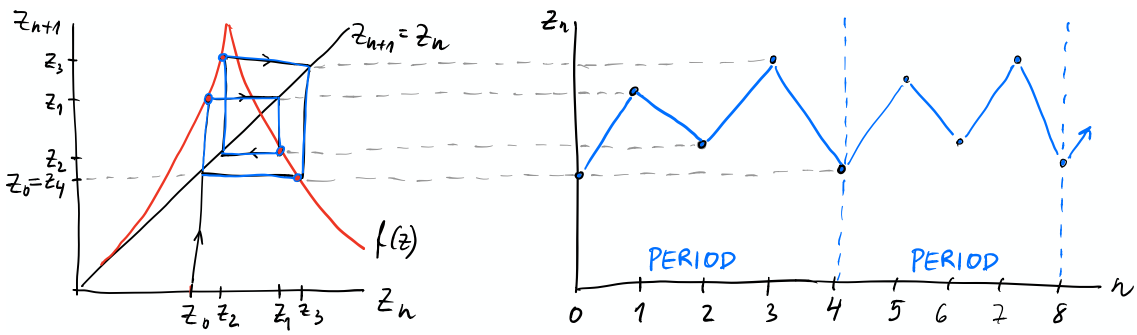

Figure 5: (Left) Period-4 orbit, shown with the blue graph, where $z_{n+4} = z_n$. (Right) The Lorenz map iterates z_n where $n \in [0, 8]$ corresponding to the cobweb diagram shown on the left. The map iterated values repeat every four iterates.

So, can trajectories in a cobweb diagram of the Lorenz map close onto themselves? We could imagine a trajectory of the Lorenz map's cobweb plot closing onto itself in a manner shown in Fig. 5. Visual inspection of the cobweb diagram shown in Fig. 5 reveals that the iterates z_n (local maxima of the three-dimensional Lorenz flow) repeat themselves such that $z_4 = z_0$ or more generally

$$z_{n+4} = z_n, \quad \forall n, \tag{7}$$

if this dynamics can be shown to be possible, then it would *strongly* suggest that **a limit-cycle might be possible** in the three-dimensional Lorenz attractor. This conclusion can be generalised further:

$$z_{n+p} = z_n, \quad \forall n, \tag{8}$$

where $p \in \mathbb{Z}^+$ is the *period* of a limit-cycle. This type of fixed point is called the **period-p point** and it represents the **period-p orbit** of the map. Period-p points are a **new type of fixed points** or in other words a new type of dynamics. A period-1 point coincides with our old friend—fixed point z^*. The period-p points where $p > 1$ don't have corresponding analogies in one-dimensional continuous systems in the manner as was discussed and shown in Sec. 1.2. This is because the period-p points represent oscillatory limit-cycle solutions which are not possible in one-dimensional systems, *cf.* Fig. 3 (see Lecture 2: Impossibility of oscillations in 1-D systems).

It is natural to assume that period-p points or orbits, very much like fixed points z^* (a period-1 point), can be either **stable (attracting trajectories) or unstable (repelling trajectories)**. If we can show that stable period-p orbits are possible in the Lorenz map, then that would *strongly* suggest a possibility of periodic solutions in the continuous-time Lorenz attractor.

Let's find the analytical expression for the relationship between z_{n+p} and z_n in (8). The n-th iterate of a selected initial condition z_0

$$z_n = \underbrace{f(f(f(\ldots f(z_0)}_{n \text{ times}}\ldots))) \equiv f^n(z_0), \tag{9}$$

where f^n is the n-th iterate map—the map applied to itself n times. Don't confuse this notation with raising the map function to the n-th power. The subsequent iterates of the closed period-p orbits, similar to the one shown in Fig. 5 with the blue trajectory, expressed analytically are

$$\begin{aligned}
z_1 &= f(z_0) \equiv f^1(z_0)\\
z_2 &= f(z_1) = f(f(z_0)) \equiv f^2(z_0)\\
z_3 &= f(z_2) = f(f(f(z_0))) \equiv f^3(z_0)\\
z_4 &= f(z_3) = f(f(f(f(z_0))) \equiv f^4(z_0)\\
&\vdots\\
z_{n+p} &= f^p(z_n),
\end{aligned} \tag{10}$$

where z is the period-p point in the period-p orbit and f^p is the p-th iterate map.

Definition: z is a **period-p point** if equation

$$\boxed{f^p(z) = z,} \tag{11}$$

where p is minimal, is satisfied.

The stability of a period-p point is determined via linearisation. For simplicity we consider stability of period-2 point

$$f^2(z) \equiv f(f(z)) = z. \tag{12}$$

Note that z is a period-2 point for map f but a fixed point (a period-1 point) for map f^2. In the previous lecture we showed that the stability of fixed point z^* depends on the slope of the map evaluated at that point. Small perturbations $|\eta| \ll 1$ evolve according to

$$\boxed{\eta_{n+1} \approx |f'(z^*)|\eta_n.} \tag{13}$$

We find

$$|(f^2(z))'| = \frac{\mathrm{d}}{\mathrm{d}z}|f(f(z))| = \begin{bmatrix}\text{chain rule } +\\ \text{orbit points}\\ z_{n+2} = z_n,\\ \text{and we select } z_0\end{bmatrix} = |f'(\underbrace{f(z_0)}_{z_1}) \cdot f'(z_0)| = |f'(z_1)| \cdot |f'(z_0)| > 1, \tag{14}$$

because for the Lorenz map $|f'(z)| > 1,\ \forall z$ in the map basin. The period-2 point is thus **unstable**. The evolution of perturbations defined by (13) generalised to all period-p points in any closed period-p orbit are the following:

$$\eta_{n+p} \approx \left|\prod_{k=0}^{p-1} f'(z_{n+k})\right| \eta_n, \tag{15}$$

here, again, by the Lorenz map property:

$$\left|\prod_{k=0}^{p-1} f'(z_{n+k})\right| > 1. \tag{16}$$

Thus, **all period-p points are unstable**. The above analysis of the Lorenz map has strongly demonstrated (not proven) that periodic solutions of the Lorenz system are not possible and that the flow is indeed **long-term aperiodic**.

2 A proper introduction to 1-D maps

This section deals with a new class of dynamical systems, introduced in Lecture 9 and used up to now without a proper introduction. In maps *time* is **discrete**, rather than continuous. These systems are known variously as **recursion relations, iterated maps, or simply maps**. The Lorenz map is such a system. When we say "map," do we mean the function f or the **recursion relation** in the following form:

$$x_{n+1} = f(x_n)? \tag{17}$$

Following common usage, we'll call both of them maps. If you're disturbed by this, you must be a pure mathematician... or should consider becoming one! Fixed point x^* of one-dimensional map (17) satisfies Eq. (3) and period-p point x satisfies Eq. (11) for minimal p.

Maps arise in various ways:

1. **As tools for analysing differential equations.** We have already encountered maps in this role. For instance, the Lorenz map provided evidence that the Lorenz attractor is aperiodic, and is not just a long-period limit-cycle. In future lectures the **Poincaré maps** will allowed us to prove the existence of a periodic solutions, and to analyse the stability of periodic solutions in general. Maps will prove to be superb tools for studying and analysing chaotic systems.
2. **As models of natural phenomena.** In some scientific contexts it is natural to regard time as discrete. This is the case in digital electronics, in parts of economics and finance theory, in impulsively driven mechanical systems, and in the study of certain animal populations where successive generations do not overlap, i.e., where a generation lives only for one season.
3. **As simple examples of chaos.** Maps are interesting to study in their own right, as mathematical laboratories for chaos. Indeed, maps are capable of much wilder behaviour than differential equations because iterates x_n *hop* along their orbits or trajectories rather than flow continuously. Continuity is very much a restriction on possible dynamics, *cf.* Lecture 6: The Poincaré-Bendixson theorem.

3 The logistic map

In a fascinating and influential review article (linked below), Robert May (1976–2020) emphasised that even simple nonlinear maps could have very complicated dynamics. May illustrated his point with the **logistic map** given by:

$$x_{n+1} = rx_n(1 - x_n), \tag{18}$$

a discrete-time analog of the **logistic equation** for population growth, where x_n is the dimensionless measure of the population in the n-th generation and parameter r is the intrinsic growth rate. As shown in Fig. 6, the graph of map (18) is a parabola with a maximum value of $r/4$ at $x = 1/2$. We restrict the control parameter r to the range $(-2$ or$)$ $0 \le r \le 4$ so that Eq. (18) maps the interval $0 \le x \le 1$ into itself.

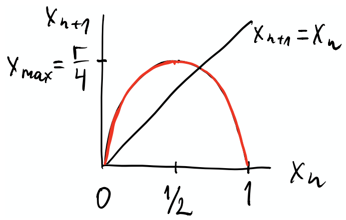

Figure 6: The logistic map shown with the red curve and the diagonal where $x_{n+1} = x_n$.

SLIDE: 3

The logistic map

The logistic map[1] has the following form:

$$x_{n+1} = r x_n (1 - x_n), \quad x_0 \in [0, 1], \quad r \in [0, 4], \quad n \in \mathbb{Z}^+, \tag{1}$$

where r is the control parameter.

Read: Robert M. May, "Simple mathematical models with very complicated dynamics," *Nature* **261**, pp. 459–467, 1976. doi:10.1038/261459a0

[1]See Mathematica .nb file (cobweb diagram and orbit diagram) uploaded to the course webpage.

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Reading suggestion

Link	File name	Citation
Paper#1	paper2.pdf	Robert M. May, "Simple mathematical models with very complicated dynamics," *Nature*, **261**, pp. 459–467, (1976). doi:10.1038/261459a0

3.1 The Lyapunov exponent

A precise estimation or calculation of the Lyapunov exponents of differential equations is not a trivial task. In the case of iterated maps it is much easier. We remind that the positive Lyapunov exponent λ is a sign of chaos, see Lecture 9, Sec. 2.

SLIDES: 4–6

The Lyapunov exponent of the logistic map

Chaos is characterised by **sensitive dependence on initial conditions**. If we take two close-by initial conditions, say x_0 and $y_0 = x_0 + \eta$ with $\eta \ll 1$, and iterate them under the map, then the difference between the two time series $\eta_n = y_n - x_n$ should grow exponentially

$$|\eta_n| \sim |\eta_0 e^{\lambda n}|, \tag{2}$$

where λ is the Lyapunov exponent. For maps, this definition leads to a very simple way of measuring the Lyapunov exponents. Solving (2) for λ yields:

$$\lambda = \frac{1}{n} \ln \left| \frac{\eta_n}{\eta_0} \right|. \tag{3}$$

By definition $\eta_n = f^n(x_0 + \eta_0) - f^n(x_0)$. Thus

$$\lambda = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \eta_0) - f^n(x_0)}{\eta_0} \right|. \tag{4}$$

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The Lyapunov exponent of the logistic map

For small values of η_0, the quantity inside the absolute value signs is just the derivative of f^n with respect to x evaluated at $x = x_0$:

$$\lambda = \frac{1}{n} \ln \left| \frac{\mathrm{d} f^n}{\mathrm{d} x} \right|_{x = x_0}. \tag{5}$$

Since, $f^n(x) = f(f(f(\ldots f(x))) \ldots)$, by the chain rule:

$$\begin{aligned} \left| \frac{\mathrm{d} f^n}{\mathrm{d} x} \right|_{x = x_0} &= \left| f'(f^{n-1}(x_0)) \cdot f'(f^{n-2}(x_0)) \cdot \ldots \cdot f'(x_0) \right| \\ &= \left| f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot \ldots \cdot f'(x_0) \right| = \left| \prod_{i=0}^{n-1} f'(x_i) \right|. \end{aligned} \tag{6}$$

Our expression for the Lyapunov exponent takes the form:

$$\lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{7}$$

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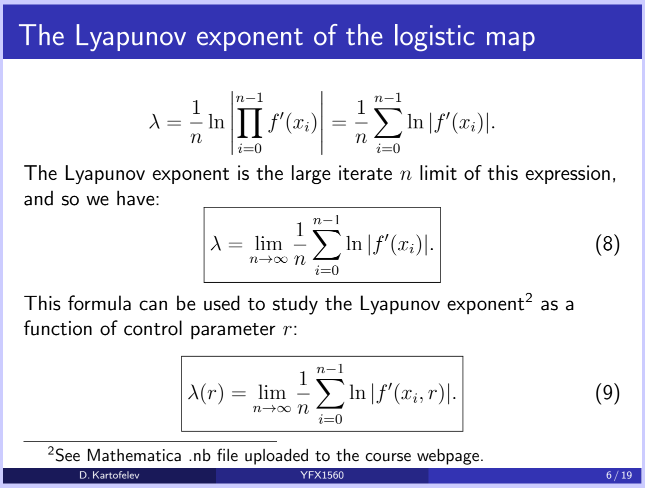


The obtained algorithm is used in the following numerical file to calculate the Lyapunov exponent of the logistic map as a function of the growth rate r.

Numerics: nb#1

Cobweb diagram and iterates of the (generalised) Lorenz map. Lyapunov exponent $\lambda(r)$ of the logistic map.

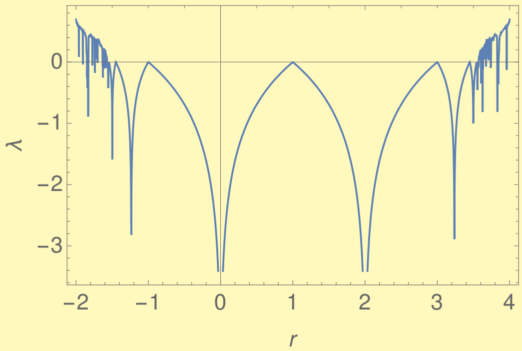


Figure 7: The Lyapunov exponent of the logistic map as a function of the growth rate r. Calculation uses 5000 iterations for every r value plotted.

3.2 Bifurcation analysis and period doubling bifurcation

Next, let's consider only the stable fixed point and stable period-p points as we incrementally increase the value of control parameter $r \geq 0$. We do that in order to simplify our analysis and to save some lecture time. The stable fixed point (a period-1 point) of the logistic map given by (18) and satisfying condition (3) is the following:

$$f(x^*) = x^* \quad \Rightarrow \quad rx^*(1-x^*) = x^* \quad |\div x^*, \tag{19}$$
$$r(1-x^*) = 1, \tag{20}$$
$$r - rx^* = 1 \quad |\div r, \tag{21}$$
$$1 - x^* = \frac{1}{r}, \tag{22}$$
$$x^* = 1 - \frac{1}{r}. \tag{23}$$

Additionally, there are the trivial solutions $x^* = 0$ and $x^* = 1$ (for initial condition $x_0 = x^* = 1$, and for $n \geq 1$, $x_n \to x^* = 0$). Fixed point (23) is stable for $|f'(x^*)| < 1$. Using map definition (18) we write

$$|f'(x^*)| = \left|[rx(1-x)]'\right|_{x=x^*} = |r - 2rx^*| < 1. \tag{24}$$

Using condition (24) with the trivial fixed point $x^* = 0$ gives us

$$|r - 0| < 1, \tag{25}$$

$$|r| < 1. \tag{26}$$

The other trivial fixed point $x^* = 1$ yields the *same* result

$$|r - 2r| < 1, \tag{27}$$

$$|-r| < 1. \tag{28}$$

Thus, the fixed points $x^* = 0$ and $x^* = 1$ are stable for $|r| < 1$. For the non-trivial fixed point $x^* = 1 - 1/r$ and for condition (24) we find

$$\left| r - 2r\left(1 - \frac{1}{r}\right)\right| < 1, \tag{29}$$

$$|r - 2r + 2| < 1, \tag{30}$$

$$|2 - r| < 1, \tag{31}$$

$$-1 < 2 - r < 1 \quad | -2, \tag{32}$$

$$-3 < -r < -1 \quad | \cdot (-1), \tag{33}$$

$$1 < r < 3. \tag{34}$$

The fixed points $x^* = 1 - 1/r$ exist and is stable for $1 < r < 3$.

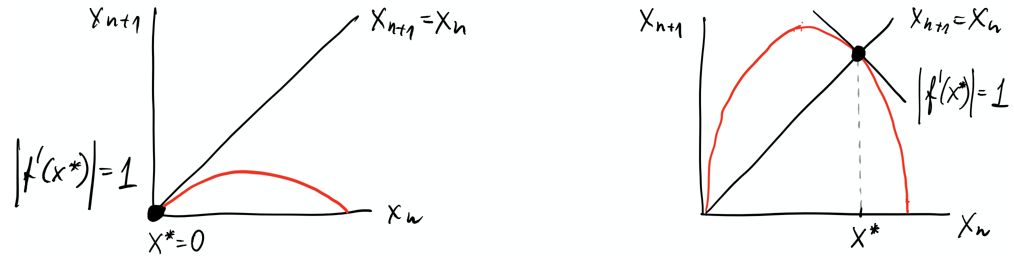


Figure 8: (Left) Map slope at the fixed points $x^* = 1 - 1/r$ (same as the trivial cases $x^* = 0 = 1$) for $r = 1$. (Right) Map slope at the fixed points $x^* = 1 - 1/r$ for $r = 3$.

It seems that the found intervals (26), (28) and (34) excluded $r = 1$ (obviously it does not satisfy $|f'(x^*)| < 1$). Let's find the value of the map slope $|f'(x^*)|$ for $r = 1$ using (24) in the case of the trivial solutions $x^* = 0$

$$|f'(x^*)| = |1 - 0| = 1, \tag{35}$$

$x^* = 1$

$$|f'(x^*)| = |1 - 2| = 1, \tag{36}$$

and in the non-trivial case for $x^* = 1 - 1/r = 1 - 1 = 0$. Which should obviously generate the same result

$$|f'(x^*)| = \left|1 - 2\left(1 - \frac{1}{1}\right)\right| = |1 - 0| = 1. \tag{37}$$

Below, it will also be beneficial to know what happens for $r = 3$, the r value just after the interval (34). We consider the non-trivial fixed point $x^* = 1 - 1/r$ and find the slope

$$|f'(x^*)| = \left|3 - 2 \cdot 3\left(1 - \frac{1}{3}\right)\right| = |3 - 4| = |-1| = 1. \tag{38}$$

Usually, slope $|f'(x^*)| = 1$ corresponds to the **period doubling or flip bifurcation point** (dynamics explained below). Values $r = 1$ and $r = 3$ are the bifurcation points. Figure 8 shows the map and map slopes $|f'(x^*)|$ evaluated at the non-trivial fixed point $x^* = 1 - 1/r$ for $r = 1$ and $r = 3$. The following interactive numerical file show the dynamics of the logistic map for $0 \le r < 1$ and $1 < r < 3$.

Numerics: nb#2

Cobweb diagram of the logistic map. Orbit diagram of the logistic map. Period doubling bifurcation.

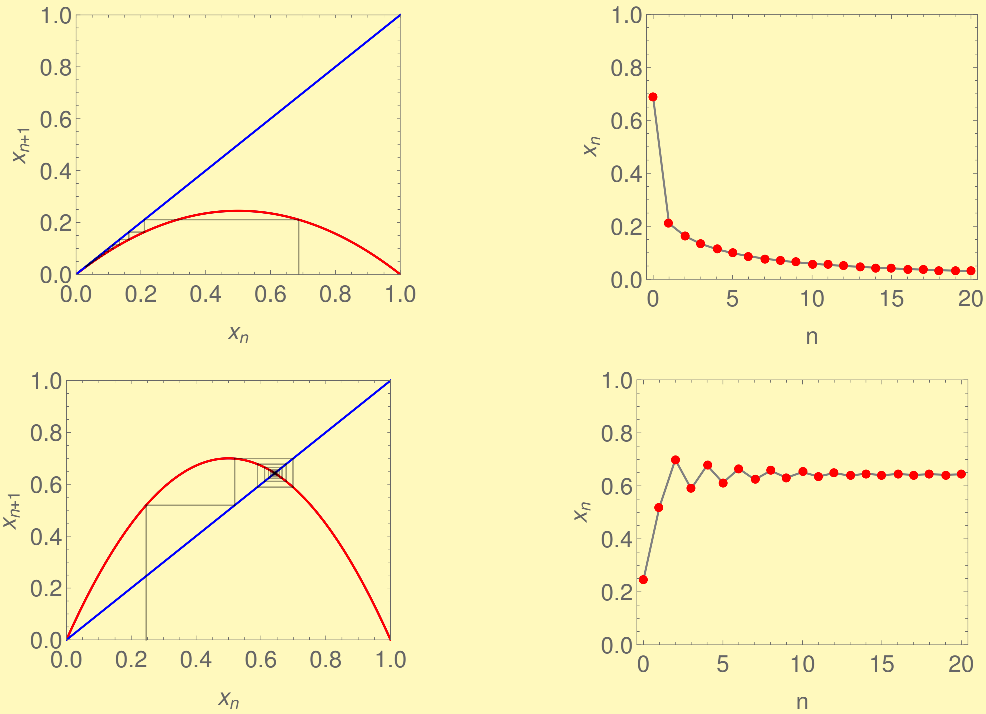


Figure 9: (Top-left) Cobweb diagram of the logistic map shown for $r = 0.98 < 1$. The fixed point $x^* = 0$. (Top-right) Map iterates corresponding to the cobweb diagram shown on the left. (Bottom-left) Cobweb diagram of the logistic map shown for interval $1 < r < 3$ with $r = 2.82$. The fixed point $x^* = 1 - 1/r$. (Bottom-right) Map iterates corresponding to the cobweb diagram shown on the left.

What happens for $r \geq 3$? What happens after the **period doubling** or **flip bifurcation** at $r = 3$? The name "flip" refers to the fact that the map trajectories start to *flip* between two values—the period-2 points in a period-2 orbit. Let's see this dynamics play out using a computer.

Numerics: nb#2

Cobweb diagram of the logistic map. Orbit diagram of the logistic map. Period doubling bifurcation.

After the **initial transient behaviour** has decayed the dynamics of the map settles to a **stable period-2 orbit**. It can be showed that period-2 orbits exist for $3 \leq r < 1 + \sqrt{6}$.

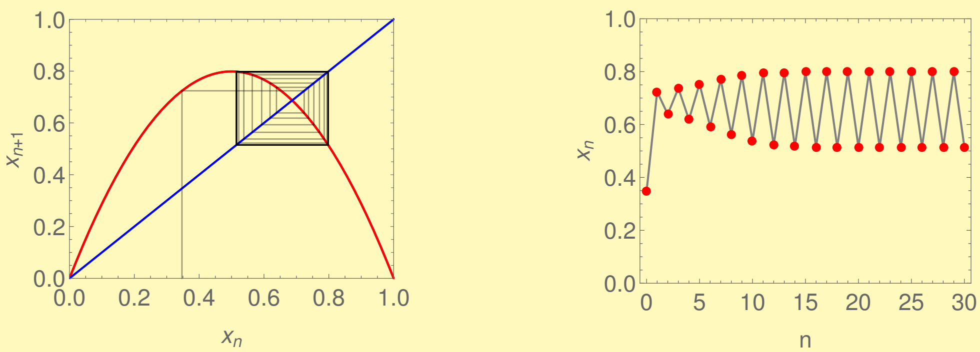


Figure 10: (Left) Cobweb diagram of the logistic map shown for $r = 3.19 \geq 3$ featuring the stable period-2 point. (Right) Map iterates corresponding to the cobweb diagram shown on the left.

Let's try to think about and unpack this outcome some more...

SLIDES: 7–9

The logistic map, period-2 window

Period-2 window for $3 \le r < 1+\sqrt{6}$.

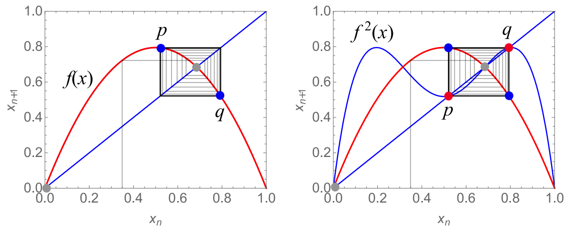


Figure: The logistic map where $r = 3.18$ and $x_0 = 0.35$. Fixed points (f.p.s) in the case where $r < 3$ are shown with the grey bullets. Period-2 points of $f(x)$ map for $r \geq 3$ are shown with the blue bullets. The fixed points of $f^2(x)$ map for $r \geq 3$ are shown with the red bullets.

The logistic map, period-2 window

Period-2 window for $3 \le r < 1+\sqrt{6}$.

$$\begin{cases} f(p) = rp(1-p) = q, \\ f(q) = rq(1-q) = p, \end{cases} \tag{10}$$

here period-2 point values p and q are the f.p.s of $f(x)$ map.

On the other hand it also holds

$$\begin{cases} f(p) = f(f(q)) \equiv f^2(q) = r[rq(1-q)][1-(rq(1-q))] = q, \\ f(q) = f(f(p)) \equiv f^2(p) = r[rp(1-p)][1-(rp(1-p))] = p, \end{cases} \Rightarrow \tag{11}$$

$$\Rightarrow \quad f^2(x) = r[rx(1-x)][1-(rx(1-x))] = x, \tag{12}$$

where period-2 point values p and q are the f.p.s of $f^2(x)$ map.

If you want to study the dynamics of the attractors of second iterate map f^2 analytically consider one of the equations in (11, slide numbering). The fourth order polynomial defined by Eq. (11)

$$r[rx(1-x)][1-(rx(1-x))] = x, \tag{39}$$

$$-r^3x^4 + 2r^3x^3 - r^2(1+r)x^2 + r^2x = x. \tag{40}$$

Intersections between the graph of second iterate map f^2 and the diagonal correspond to the solutions of $f^2(x) = x$ (40).

Stability of f.p.s of f^2 map in period-2 orbit

We need to know the slopes of period-2 points

$$\begin{cases} f(p) = rp(1-p) = q, \\ f(q) = rq(1-q) = p. \end{cases}$$

According to the chain rule it holds that

$$(f^2(x))' \equiv (f(f(x))' = f'(f(x)) \cdot f'(x). \tag{13}$$

In our case:

$$\left.\begin{aligned} (f^2(p))' &= f'(f(p)) \cdot f'(p) = f'(q) \cdot f'(p) \\ (f^2(q))' &= f'(f(q)) \cdot f'(q) = f'(p) \cdot f'(q) \end{aligned}\right\} \Rightarrow (f^2(p))' = (f^2(q))'. \tag{14}$$

Above follows from the commutative property of multiplication.

The slopes of map f^2 at its fixed points p and q are equal and they are the products of map f slopes at its respective period-2 points.

What happens for $r \geq 1+\sqrt{6}$? Once again we use a computer...

NUMERICS: NB#2

Cobweb diagram of the logistic map. Orbit diagram of the logistic map. Period doubling bifurcation.

A **period-4 orbit** has emerged. It can be shown that period-4 orbits exist for $1+\sqrt{6} \leq r < 3.54409...$

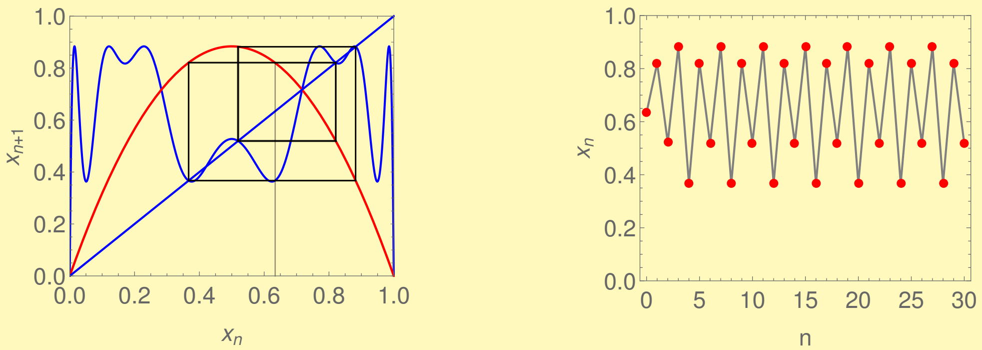


Figure 11: (Left) Cobweb diagram of the logistic map shown for $r = 3.535 \geq 1 + \sqrt{6}$ featuring the stable period-4 point. The blue graph shows the fourth iterate map f^4. (Right) Map iterates corresponding to the cobweb diagram of map f shown on the left.

Figure 12 counts the period-4 points or iterate values in the period-4 orbit of the logistic map. Compare Fig. 12 to Figs 5 and 11.

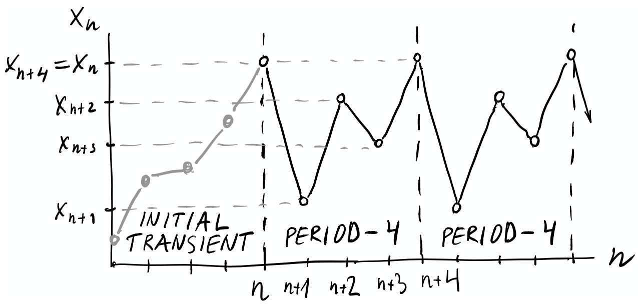
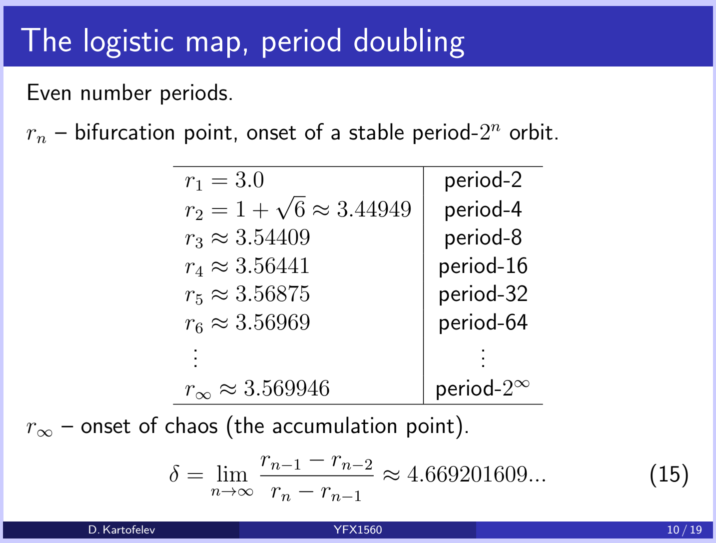


Figure 12: Period-4 orbit with its repeated iterates as they should appear in the logistic map.

What have we seen so far? As we have incrementally increased the value of control parameter r the periods of the fixed points have increased as well—(from period-1) to period-2 and finally to period-4. **The period is clearly doubling** starting from period-2. This type of bifurcation is called the **period doubling bifurcation**. Now the name "period doubling" introduced above alongside **"flip" bifurcation** should make more sense. Further period-doublings to orbits of period-8, -16, -32, -64, etc., also occur as r is increased.

SLIDE: 10

Specifically, let r_n denote the value of r where a stable 2^n-orbit first appears. Then computer experiments reveal that:



The logistic map, period doubling

Even number periods.

r_n – bifurcation point, onset of a stable period-2^n orbit.

$r_1 = 3.0$	period-2
$r_2 = 1 + \sqrt{6} \approx 3.44949$	period-4
$r_3 \approx 3.54409$	period-8
$r_4 \approx 3.56441$	period-16
$r_5 \approx 3.56875$	period-32
$r_6 \approx 3.56969$	period-64
$\vdots$	$\vdots$
$r_\infty \approx 3.569946$	period-2^∞

r_∞ – onset of chaos (the accumulation point).

$$\delta = \lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609... \qquad (15)$$

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Note that the successive bifurcations come faster and faster. Ultimately, the r_n converge to a limiting value r_∞. The convergence is (essentially) geometric (it's geometric near r_∞): in the limit of large n, the distance between successive transitions shrinks by a constant factor of δ. This ratio is called **the Feigenbaum constant**. After the **accumulation point** $r > r_\infty$ the dynamics becomes **chaotic** as we'll see below.

3.3 Orbit diagram

The orbit diagram is also called the **fig tree diagram** (Feigenbaum in German means "fig tree") or *incorrectly* the **Feigenbaum diagram**. You might guess that the system would become *more and more chaotic* for $r > r_\infty$ as r increases, but in fact the dynamics are more subtle than that. To see the **long-term behaviour for all values of** r **at once**, we plot the **orbit diagram** a special kind of bifurcation diagram. Orbit diagram plots the system's attractor (stable fixed points and stable period-p points) as a function of control parameter r.

To generate the orbit diagram for yourself, you'll need to write a computer program with two loops. First, choose a value of r. Then generate an orbit starting from some random initial condition x_0 Iterate for 300 cycles or so, to allow the system to settle down to its eventual stable behaviour. Once the transients have decayed, plot many points, say $x_{301}, \ldots, x_{900}$ above that r. Then move to an adjacent value of r and repeat, eventually sweeping across the whole diagram. The following interactive numerical file shows the orbit diagram for the logistic map.

Numerics: nb#2

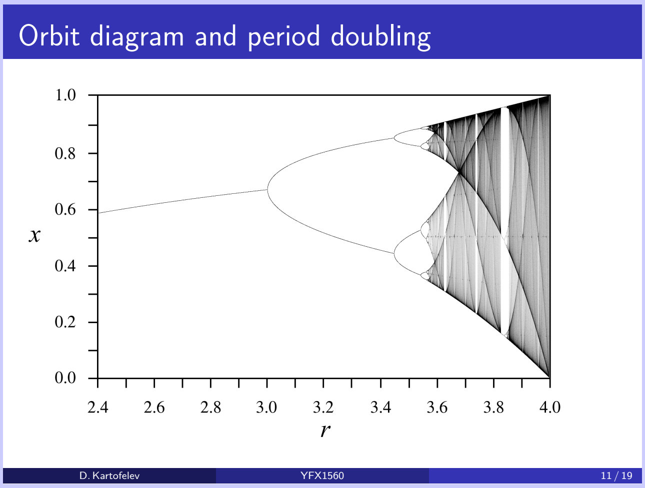
Cobweb diagram of the logistic map. Orbit diagram of the logistic map. Period doubling bifurcation.

The connection between the cobweb diagram, orbit diagram and the Lyapunov exponent is shown in the following interactive numerical file.

Numerics: nb#3

The logistic map: cobweb diagram, orbit diagram and map iterates, the Lyapunov exponent.

Slides: 11, 12



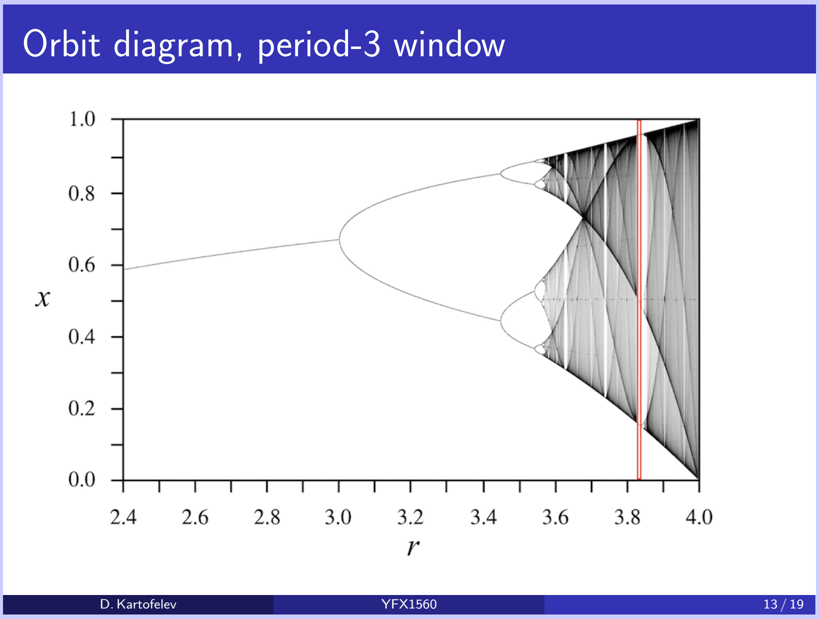
The **period doubling** is driven by the subsequent **flip bifurcations or supercritical pitchfork bifurcations** (if we use the nomenclature introduced in Lecture 2). As mentioned above, unstable fixed point and period-p points are omitted from the **orbit diagram**. The **Feigenbaum diagram** shows both stable and unstable fixed points and period-p points or simply diagram branches (not shown here).

The orbit diagram of the logistic map features **self-similarity**. In mathematics, a **self-similar** object is (exactly or) approximately similar to a part of itself, i.e., the whole has the same shape or quality as one or more of its sub-parts.

3.4 Tangent bifurcation and odd number period-p points

Can an odd number period appear in the logistic map—e.g., a period-3 point? One of the most intriguing features of the orbit diagram is the occurrence of **periodic windows** for $r > r_\infty$.

SLIDE: 13



The period-3 window that occurs for $1+\sqrt{8} \le r \le 3.8415...$ is the most conspicuous. On the above slide it is shown within the red rectangle. Suddenly, against a backdrop of chaos, a stable period-3 orbit appears out of the blue.

Let's see if we can find the period-3 orbit and window of the logistic map using a computer.

NUMERICS: NB#2

Cobweb diagram of the logistic map. Orbit diagram of the logistic map. Period doubling bifurcation.

The **tangent bifurcation** is a new type of bifurcation for us, that occurs at $r = 1+\sqrt{8}$.

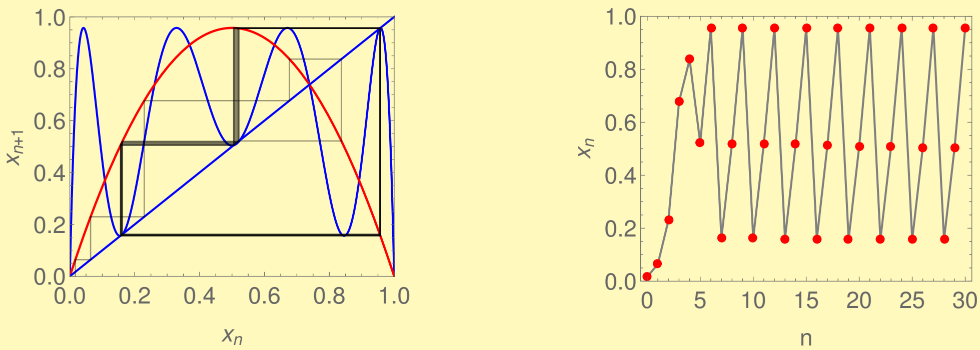


Figure 13: (Left) Period-3 orbit of the logistic map can be found in the period-3 window of the orbit diagram, see Slide 11. Here, showing the trajectory for $r = 3.83$ and $x_0 \gtrsim 0$. (Right) Map iterates x_n corresponding to the cobweb diagram of the logistic map shown on the left.

The mechanism responsible for the occurrence of the **odd number periods-p points** is shown in Fig. 14. The intersections between the graph of f^3, shown in red, and the diagonal correspond to solutions of $f^3(x) = x$. There are eight solutions for $r \ge 1+\sqrt{8}$, six of interest to us are marked with the smaller blue dots, and two imposters that are not genuine period-3; they are actually fixed points, or period-1 points for which $f(x^*) = x^*$. The blue filled dots in Fig. 14 correspond to a stable period-3 cycle; note that the slope of $f^3(x)$ is shallow at these points, consistent with the stability of the cycle. In contrast, the slope exceeds one at the cycle marked by the empty blue dots; this period-3 orbit is therefore unstable.

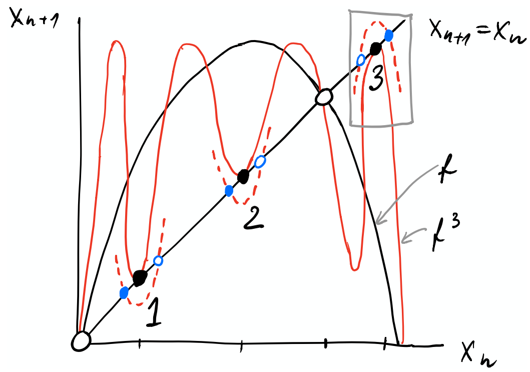


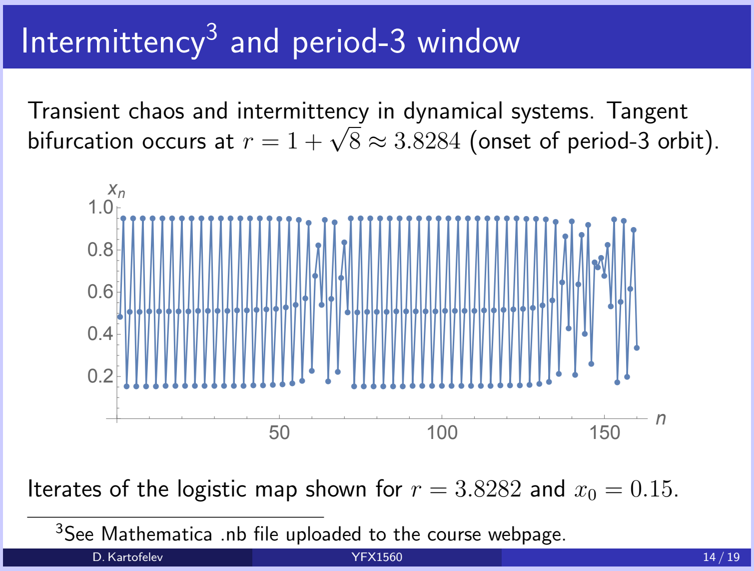
Figure 14: A period-3 orbit can be found in the period-3 window of the orbit diagram. The hollow bullets correspond to the unstable period-p points, the filled bullets to the stable ones. The dashed and continuous curves shown within the grey rectangular outline is a smaller distorted copy of map f in its entirety.

Now suppose we decrease r toward the chaotic regime $r < 1 + \sqrt{8}$. Then the red dashed graph in Fig. 14 changes shape—the hill moves down and the valleys rise up. The curve therefore moves towards the diagonal. Figure 14 shows that when $r = 1 + \sqrt{8}$, the six blue intersections have merged to form three black filled period-3 points by **becoming tangent to the diagonal**. At this critical value of r, the stable and unstable period-3 cycles coalesce and annihilate in the **tangent bifurcation**. This transition defines the beginning of the period-3 window discussed above.

All odd number periods will also undergo the period doubling. This means that all number periods $[1, \infty)$ are eventually represented in the orbit diagram.

Figure 14 also explains the **self-similarity** present in the orbit diagram as shown on Slide 12 for different scales of magnification. The peaks (shown within the grey rectangle) and valleys of map f^3 are smaller distorted copies of the original map f. Thus, the local dynamics of the cobweb trajectories for map f^3 must be similar to the dynamics of the original map f.

Slide: 14



The dynamics closely related to the tangent bifurcation is **intermittency**. The logistic map exhibits **intermittent chaos** for r values just before the **period-3 window**, i.e., just before the onset of the period-3 orbit.

The following numerical file was used to calculate and create the graph shown on the above slide.

Numerics: nb#4

Period-3 window and intermittency in the logistic map.

4 Sine map and universality of period doubling

It can be shown that in **all unimodal maps** same dynamics of period doubling occurs. For example we consider the sine map:

SLIDE: 15

Universality of period doubling in unimodal maps

1-D sine map[4]. The sine map has the form:

$$x_{n+1} = r\sin(\pi x_n), \quad x_0 \in [0,1], \quad r \in [0,1], \quad n \in \mathbb{Z}^+, \tag{16}$$

where r is the control parameter.

Read: Mitchell J. Feigenbaum, "Quantitative universality for a class of nonlinear transformations," *Journal of Statistical Physics* **19**(1), pp. 25–52, 1978, doi:10.1007/BF01020332

Read: Mitchell J. Feigenbaum, "Universal behavior in nonlinear systems," *Physica D: Nonlinear Phenomena* **7**(1–3), pp. 16–39, 1983, doi:10.1016/0167-2789(83)90112-4

[4]See Mathematica .nb file (cobweb diagram and orbit diagram) uploaded to the course webpage.

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The graph of the sine map has the **same basic shape** as the graph of the logistic map. Both curves are smooth, concave down, and have a **single maximum**. Such maps are called **unimodal**.

Sine is also **a transcendental function** opposed to an algebraic function as is the polynomial that defines the logistic map. A transcendental function is an analytic function that does not satisfy a polynomial equation. In other words, a transcendental function "transcends" algebra in that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction.

The **transcendental nature** of the sine map *must* make the underlying higher-dimensional physics *represented* by the sine map **fundamentally different** from, e.g., the three-dimensional Lorenz flow sampled by the Lorenz map.

Let's study the dynamics of the sine map using a computer:

NUMERICS: NB#5

The sine map iterates, cobweb and orbit diagrams where $x_n \in [0,1]$.

In the following numerical file the basin of attraction is widened to span $[-1,1]$. In effect, here two unimodal maps are placed side-by-side. Sine map takes the following form:

$$x_{n+1} = r\sin(\pi x_n), \quad x_0 \in [-1,1], \quad r \in [0,1], \quad n \in \mathbb{Z}^+, \tag{41}$$

where r is the control parameter.

NUMERICS: NB#6

The sine map iterates, cobweb and orbit diagrams where $x_n \in [-1,1]$.

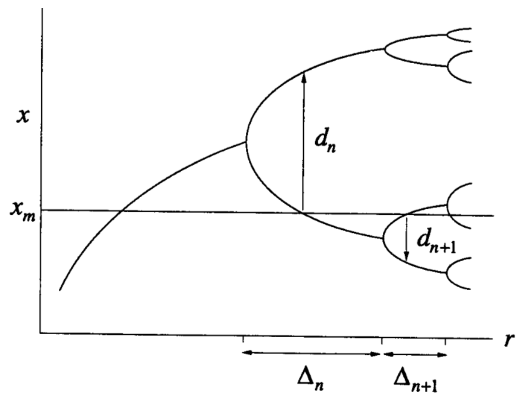
The dynamics of sine map is surprisingly similar to the dynamics of the logistic map. Mitchell J. Feigenbaum was one of the first researchers[1] to discover the **quantitative laws that are independent of unimodal map functions f**. By that we mean that the algebraic form of $f(x)$ is irrelevant, only its overall shape matters, i.e., unimodality.

The following slide shows the scaling constants that are related to these universal laws that are present in all the unimodal maps:

[1]The first published works on period doubling and related phenomena were authored by a Finnish mathematician Pekka Juhana Myrberg (1892–1976).

Slide: 16

1-D unimodal maps and the Feigenbaum constants



$$\delta = \lim_{n\to\infty} \frac{\Delta_{n-1}}{\Delta_n} = \lim_{n\to\infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609... \tag{17}$$

$$\alpha = \lim_{n\to\infty} \frac{d_{n-1}}{d_n} \approx -2.502907875... \tag{18}$$

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Here $x_m = \max f(x)$ is the maximum of the map graph. The **Feigenbaum constants** are valid up to the onset of chaos at accumulation point r_∞ and inside each **periodic window** for $r > r_\infty$.

In addition to scaling law in control parameter r direction, shown earlier, Feigenbaum also found a scaling law for the vertical x-direction of the orbit diagram. **The Feigenbaum constants** are **universal** the same convergence rate appears no matter what unimodal map is iterated! They are **mathematical constants**, as basic to period doubling as π is to circles.

Reading suggestion

Link	File name	Citation
Paper#2	paper3.pdf	Mitchell J. Feigenbaum, "Quantitative universality for a class of nonlinear transformations," *Journal of Statistical Physics*, **19**(1), pp. 25–52, (1978). doi:10.1007/BF01020332
Paper#3	paper4.pdf	Mitchell J. Feigenbaum, "Universal behavior in nonlinear systems," *Physica D: Nonlinear Phenomena*, **7**(1–3), pp. 16–39, (1983). doi:10.1016/0167-2789(83)90112-4

5 Universal route to chaos

We showed that the *qualitative* dynamics of the logistic and sine maps are **identical**. They both undergo **period doubling routes to chaos**, followed by **periodic windows** interwoven with **chaotic bands**. Even more remarkably, the periodic windows occur in the same order, and with the same relative sizes. For instance, the period-3 window is the largest in both cases, and the next largest windows preceding it are period-5 and period-6. But there are *quantitative* differences. For instance, the period doubling bifurcations occur later (for greater parameter r value) in the logistic map, and the periodic windows are thinner.

Turns out that the onset of chaos via period doubling is predominant in nature and in artificial chaotic systems. The Feigenbaum constants have real predictive power in various scientific applications. The **period doubling bifurcation** is **the "route" taken by nonlinear systems to reach chaotic solution**. In term of the bifurcations introduced in Lecture 2, the period doubling bifurcation can also be seen as a series of subsequent or succeeding supercritical pitchfork bifurcations, see Slide 11.

Revision questions

1. What is cobweb diagram?
2. What is recurrence map or recurrence relation?
3. What is 1-D map?
4. How to find fixed points of 1-D maps?
5. What is the Lorenz map?
6. What is the logistic map?
7. What is sine map?
8. What is period doubling?
9. What is period doubling bifurcation?
10. What is tangent bifurcation?
11. Do odd number periods (period-p orbits) exist in chaotic systems?
12. Do even number periods (period-p orbits) exist in chaotic systems?
13. Can maps produce transient chaos?
14. Can maps produce intermittency?
15. Can maps produce intermittent chaos?
16. What is orbit diagram (or the Feigenbaum diagram)?
17. What are the Feigenbaum constants?