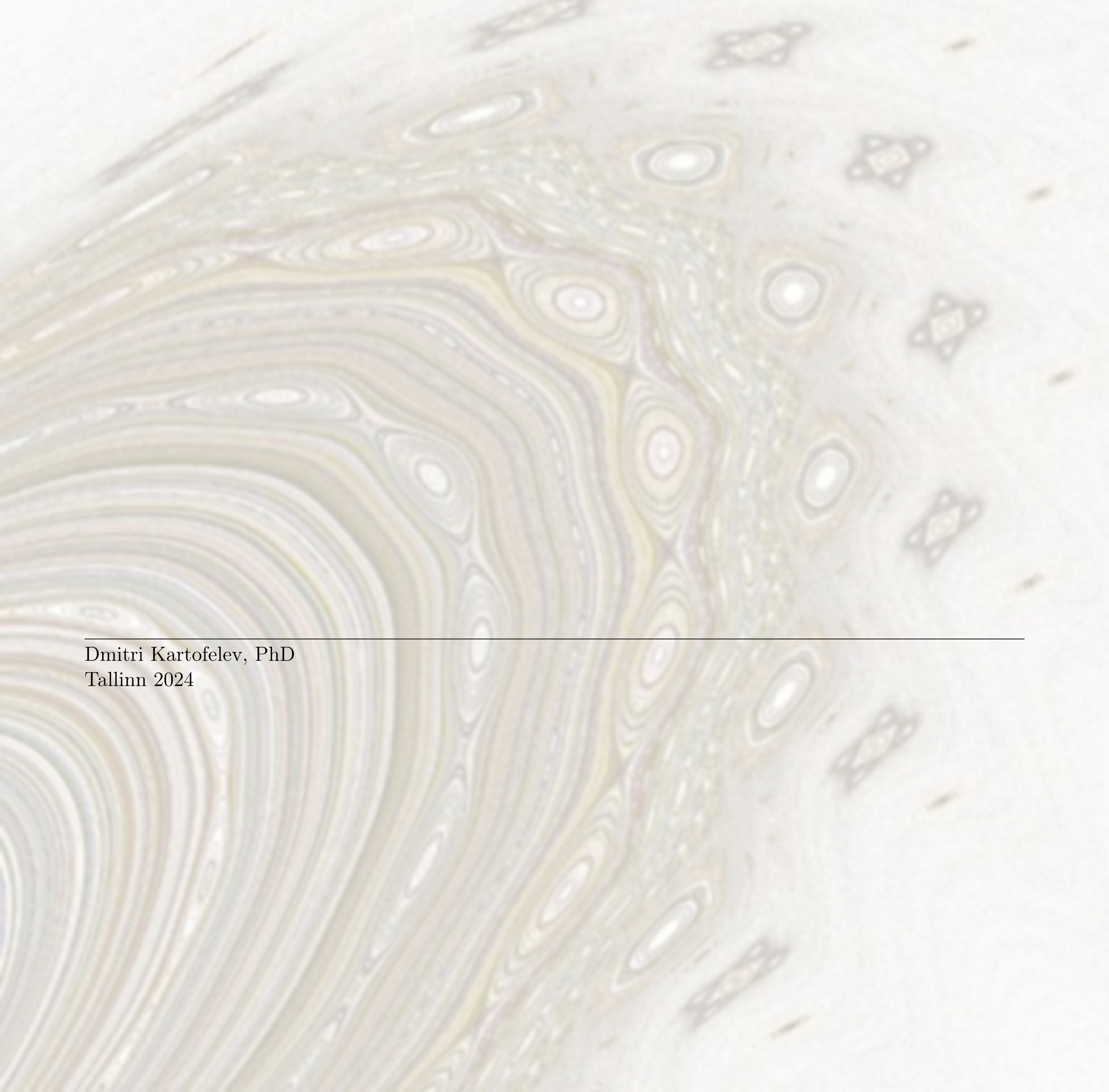
LECTURE NOTES:
Nonlinear Dynamics
YFX1560



Dmitri Kartofelev, PhD
Tallinn 2024

Lecture Notes: Nonlinear Dynamics, YFX1560
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This document is compiled using the LaTeX document preparation system.
`https://www.tud.ttu.ee/web/dmitri.kartofelev/`

IT Akadeemia HITSA

Creation of this study aid was supported by: HITSA Information Technology Foundation for Education, Project EITSA 20007, "Erialaste digipädevuste arendamine ja e-õppevara loomine täppisteadustes." (*IT Akadeemia e-õppe arendusprojekt täppisteadustes*, 2020)

Preface

Lecture notes

These lecture notes are created for the students of Tallinn University of Technology (TalTech) who are taking the Nonlinear Dynamics (YFX1560) course. These lecture notes and other resources can be accessed through the course webpage at `https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560.html`.

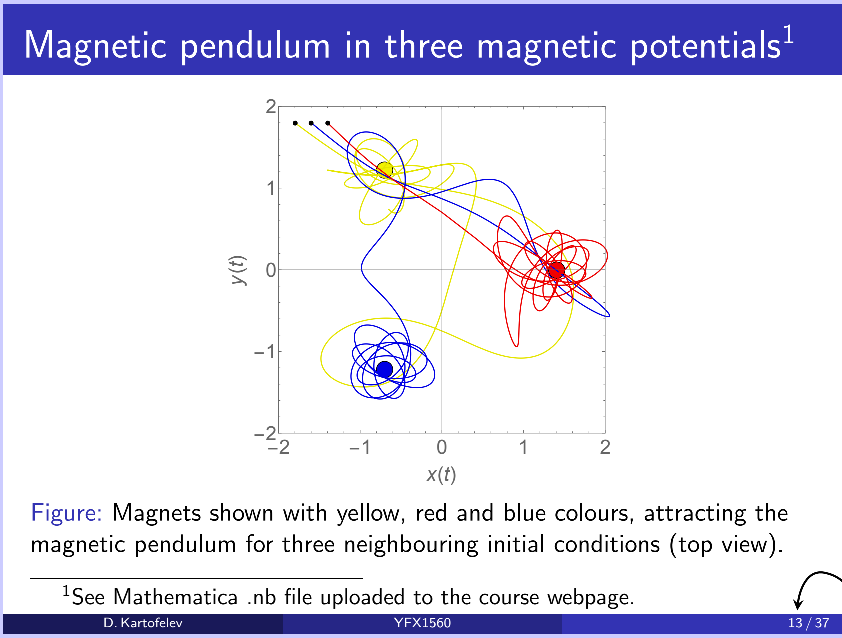
Document navigation

Blue text boxes

The *blue text boxes* are used to refer to the information on the slides or the slides themselves that are used during the lectures. Below is an example of a blue text box.

Slide: 13

Slide or multiple slides, clarifying comments, etc., related to the presented slide or slides are placed inside the blue boxes.



The upper-right corners of the blue text boxes show the slide numbering information.

Yellow text boxes

The *yellow text boxes* are used to refer to the numerical files (Wolfram Mathematica notebook files) used during the lectures. An example of a yellow text box and its contents is shown below.

Numerics: nb#1

Short description of a script/program contained in the notebook file. Same descriptions can be found in the index of the numerical files.

Additional information, clarifications, comments, calculation results etc., related to the numerical file/s are also placed inside the yellow box.

The upper-right corners of yellow text boxes feature a hyperlink referring to the numerical file itself. These numerical files can also be found on the course webpage and they are downloadable from the aforementioned index contained in this .pdf file.

Grey text boxes

The *grey text boxes* are reserved for the revision questions and they are featured at the end of each lecture note. An example of a grey text box is shown below.

Revision questions

1. What is dynamics?
2. Determine if the following equations/systems are linear or nonlinear:

$$\dot{x} = \sin x. \tag{1}$$

The same questions are featured on the slides. The revision questions for the entire course are compiled into a single .pdf file that is accessible through the course webpage.

Hand-drawn graphs and figures

Figure 1: An example of a hand-drawn graph.

The figures and graphs drawn on a black board or dry erase board during the lectures will be reproduced in these lecture notes using crude hand-drawings. An example of a hand-drawn graph is shown in Fig. 1.

Suggested reading text boxes/tables

Some lectures will feature suggested reading materials in the form of refereed academic papers or textbook chapters/excerpts. The index of the used papers can be found on the course webpage. These lecture notes will refer to suggested reading materials using transparent tables. An example table is shown below.

Link	**File name**	**Citation**
Paper#1	paper0.pdf	Evgeni E. Sel'kov, "Self-oscillations in glycolysis 1. A simple kinetic model," *European Journal of Biochemistry*, **4**(1), pp. 79–86, (1968) doi:10.1111/j.1432-1033.1968.tb00175.x

Acknowledgements

The course is based on the following textbooks:
- S.H. Strogatz, *Nonlinear Dynamics and Chaos With Applications to Physics, Biology, Chemistry, and Engineering*, Second Edition, Avalon Publishing, 2014.
- K.T. Alligood, T.D. Sauer, J.A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, 2000.
- A. Medio, M. Lines, *Nonlinear Dynamics A Primer*, Cambridge University Press, 2003
- H. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, New York: Springer-Verlag, 2004.

In some instances small sections of the above textbooks (Strogatz, and Medio and Lines) are copy and pasted directly into these course lecture notes.

LECTURE 1: *Practical course navigation*, INTRODUCTION, MAGNETIC PENDULUM, HISTORY, DYNAMICS, NONLINEARITY, ORDINARY HOMOGENEOUS DIFFERENTIAL EQUATION, 1-D PROBLEM, PHASE SPACE, PHASE PORTRAIT, FIXED POINT AND ITS STABILITY

Contents

Handout: List of lectures and the course syllabus.
Demonstration: Chaotic magnetic pendulum.

1 *Practical course navigation*

Welcome to Nonlinear Dynamics course! This course is aimed at newcomers to the topics and problems of nonlinear dynamics and chaos, especially students taking a first course in the subject.

SLIDES: 3–7

The main communication channels between the Lecturer and students are the course webpage, TalTech Moodle web page and e-mail.

General and contact info

Contact info: https://www.tud.ttu.ee/web/dmitri.kartofelev/

E-mail: dmitri.kartofelev@taltech.ee

Online resources:

- **Course webpage (coursework, lecture notes, etc.):** https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560.html
- **Moodle (course discussion forum and grades):** https://moodle.taltech.ee/course/view.php?id=21971
- **Video lectures by S. Strogatz** (author of the main textbook): https://www.youtube.com/playlist?list=PLbN57C5Zdl6j_qJA-pARJnKsmROzPnO9V

Course overview and syllabus:

- https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560/topics_covered.pdf
- https://ois2.taltech.ee/uusois/aine/YFX1560

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General and contact info

Examination: One coursework and one written exam.

- Written exam. Revision questions: https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560/revision_questions.pdf
- Two part coursework. Requirements: https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560/CW_requirements.pdf
- Coursework is a prerequisite for taking the exam.
- Course grading criteria: https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560/assessment2.pdf

Personal consultation: My office on Wednesdays and Fridays (office hours, schedule the meeting in advance)

- Visit me at least once to discuss the coursework.

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Numerical files and programs

The lecturer will be using *Wolfram Mathematica* programs/codes to demonstrate the nonlinear phenomena being discussed.

To open .nb files:

- Use you own computer/laptop and free **Wolfram Player**. Download link: https://www.wolfram.com/player/

To open and execute .nb files:

- Classroom with access to **Mathematica** software: ~~SOC-408~~
- **Wolfram Cloud:** https://www.wolframcloud.com
- Wolfram Mathematica installation

Indexes of the numerical files used:

- Index of .nb files https://www.tud.ttu.ee/web/dmitri.kartofelev/YFX1560/Index_num.pdf

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Literature and textbooks

- S.H. Strogatz, **Nonlinear Dynamics and Chaos With Applications to Physics, Biology, Chemistry, and Engineering**, Second Edition, Avalon Publishing, 2014.
  - M. Dichter, *Student Solutions Manual for Nonlinear Dynamics and Chaos*, 2nd Edition (Volume 2), CRC Press, 2018
    - *e-library access:* https://ebookcentral.proquest.com/lib/tuee/detail.action?docID=5394195
- K.T. Alligood, T.D. Sauer, J.A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, 2000.
- H. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, New York: Springer-Verlag, 2004.

- Ü. Lepik, J. Engelbrecht, *Kaoseraamat*, Teaduste Akadeemia Kirjastus, 1999.
- D.W. Jordan, P. Smith, *Nonlinear Ordinary Differential Equations*, Oxford Univ. Press, 1988.

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The main textbooks are the first three books. Most of the course is based on the Strogatz's textbook. The Strogatz's textbook and Dichter's solution manual are available at TalTech Library.

Estonian speaking student are encouraged to read the textbooks written in Estonian to familiarise themselves with the correct technical terminology both in Estonian and English languages.

Literature and textbooks

- H.G. Solari *et al.*, *Nonlinear Dynamics*, Inst. of Physics, 1996.
- R.C. Hilborn, *Chaos and Nonlinear Dynamics*, Oxford Univ. Press, 1994.
- D. Kaplan, L. Glass, *Understanding Nonlinear Dynamics*, Springer, 1995.
- J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.
- *Encyclopaedia of Nonlinear Sciences*, ed. by A. Scott, Routledge, 2005.
- A. Medio, M. Lines, *Nonlinear Dynamics: A Primer*, Cambridge University Press, 2001.

During the forthcoming weekend google topics of nonlinear dynamics and chaos.
Suggestion: BBC documentary "The secret life of chaos" (link).

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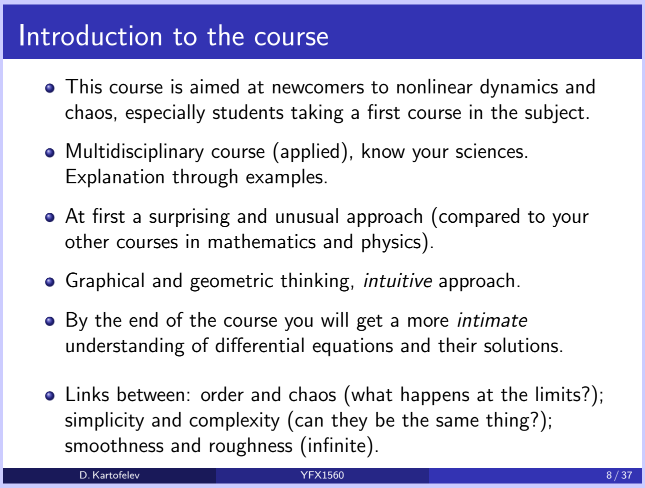
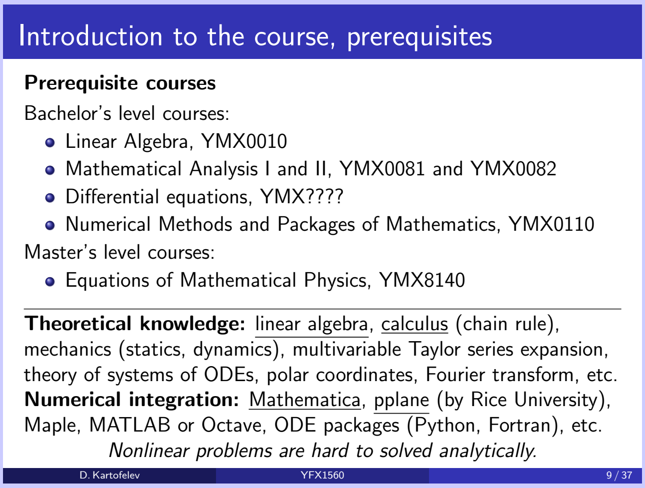
The BBC documentary can be seen as a well-worded introduction to some of the topics discussed in this course.

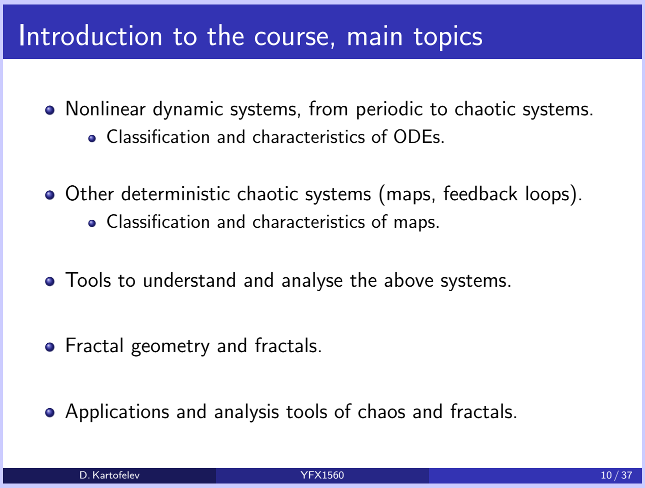
2 Introduction to the course

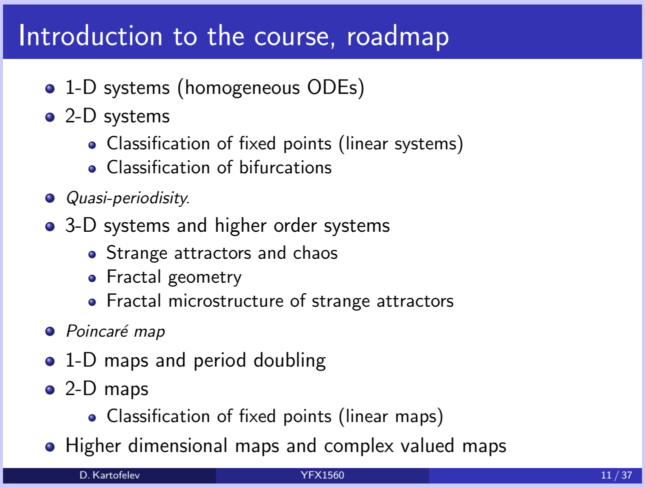
Slides: 8–11

The course can be classified as an *applied* course. We will not be spending much time on proving mathematical theorems, arguing abstract arguments, etc. Virtually every idea will be illustrated by some application to science or engineering. In some cases, the applications are drawn from the *recent* research literature.









By the "roadmap" we mean logical structure of the course. Our broader goal will be to provide you with an overview of the classification of ODEs and iterative maps based on their solution types which can include chaotic dynamics.

3 Classroom demonstration: Chaotic pendulum

Before we start with the course let's take a look at a mechanical system with a *special* type of dynamics (properties defined properly later in the course). Namely, the magnetic pendulum in three magnetic potentials a.k.a. the chaotic pendulum. The set-up is shown in Fig. 2. By the end of the course you will be able to understand, analyse and quantify this type of dynamics.

Discussion with the students: Can a physicist predict the magnet at which the pendulum stops swinging, i.e., the end state of the system for $t \to \infty$ given its initial position and initial velocity? If you think that the question is too vague, explain.

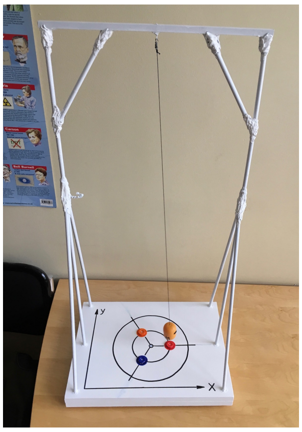


Figure 2: Magnetic pendulum in three magnetic potentials a.k.a. chaotic pendulum.

It might be surprising for you. The calculations based on an accurate physical descriptions of the proposed pendulum problem will *generally* fail to predict the end-state of the system accurately.

SLIDES: 13, 14

Using the classical mechanics and some magnetism one can model the pendulum's motion by numerically integrating the relevant equations of motion.

Magnetic pendulum in three magnetic potentials[1]

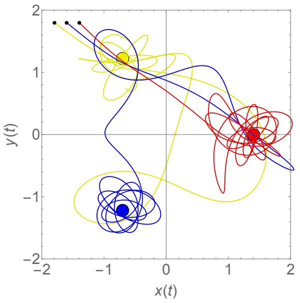


Figure: Magnets shown with yellow, red and blue colours, attracting the magnetic pendulum for three neighbouring initial conditions (top view).

[1]See Mathematica .nb file uploaded to the course webpage.

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Magnetic pendulum in three magnetic potentials

System is modeled with the following equations of motion:

$$\begin{cases} \ddot{x} + R\dot{x} - \displaystyle\sum_{i=1}^{N} \frac{x_i - x}{\left(\sqrt{(x_i - x)^2 + (y_i - y)^2 + d^2}\right)^3} + Cx = 0, \\ \ddot{y} + R\dot{y} - \displaystyle\sum_{i=1}^{N} \frac{y_i - y}{\left(\sqrt{(x_i - x)^2 + (y_i - y)^2 + d^2}\right)^3} + Cy = 0, \end{cases} \tag{1}$$

where R is proportional to the air resistance and overall attenuation, C is proportional to the effects of gravity, N is the number of magnets, the i-th magnet is positioned at (x_i, y_i), d is the distance between the pendulum at rest and the plane of magnets.

Additionally we assume that the pendulum length is long compared to the spacing of the magnets. Thus, we may assume for simplicity that the metal ball moves about on a xy-plane.

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It seems that the end state is strongly dependant on the precise initial position (x_0, y_0) of the pendulum. The initial conditions that are close (nearby) to each other do not appear to produce similar pendulum trajectories and system end states. The **predictive power** of system (1) **is not obvious**.

The numeric solution to the magnetic pendulum in three magnetic potentials is linked below.

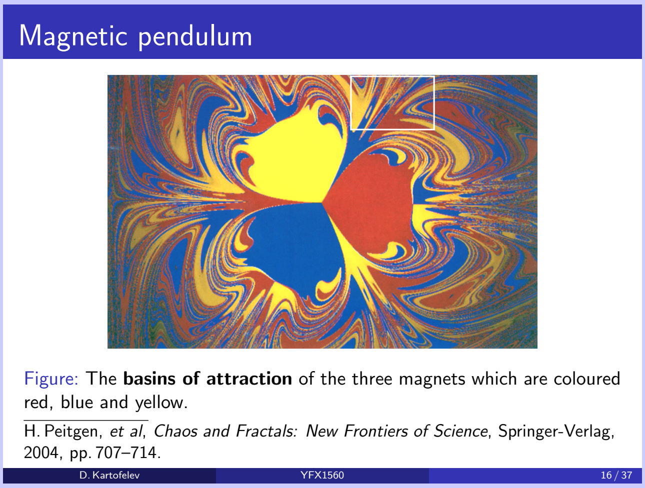
NUMERICS: NB#1

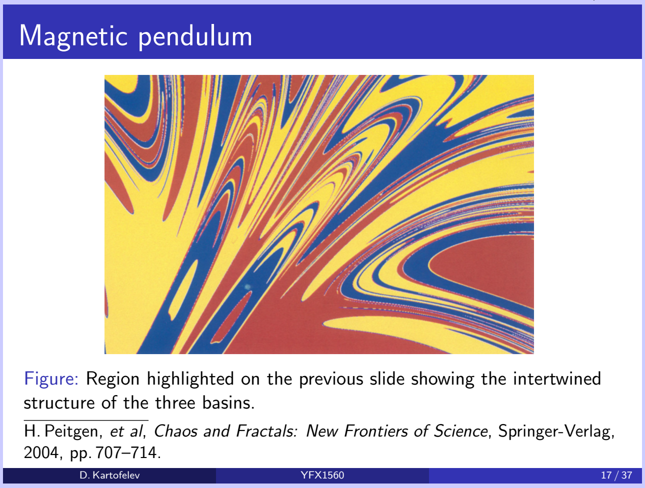
Numerical integration of the dynamical system describing magnetic pendulum's motion. A pendulum in which a metal ball (or a magnet) is attached to its end and which is oscillating over a plane where a set of attractive magnets are present (2–6 magnets).

The principal simplifying assumptions are the following:

- The pendulum length is long compared to the spacing of the magnets. Thus, we assume that the ball moves about on a plane rather than on a sphere with a large radius.
- The magnets are point attractors positioned a short distance below the pendulum plane at the vertices of an equilateral triangle.
- All possible effects related to the Eddie current driven electromagnetism caused by the supporting aluminium bars are ignored.

SLIDES: 16, 17





A closer look at this dynamics is shown here. The **basin of attraction** is defined as a set (region) of initial conditions (x_0, y_0) that end up at a specific magnet (red, yellow or blue).

Discussion with students: What do you think we would see if we further enlarged this image (increase the image resolution)? Keep in mind that two close-by initial states can produce differing results, as the above numerical simulation demonstrated.

Hopefully, this demonstration generated some open-ended questions and a genuine interest for the subject and the lectures to come.

4 History of the discipline *(skip)*

——— *Skip if needed:* start ———

The following is a shot excerpt from our main textbook (Strogatz, pp. 2–5, 1994):

The subject began in the mid-1600s, when Newton invented differential equations, discovered his laws of motion and universal gravitation, and combined them to explain Kepler's laws of planetary motion. Specifically, Newton solved the two-body problem-the problem of calculating the motion of the earth around the sun, given the inverse-square law of gravitational attraction between them. Subsequent generations of mathematicians and physicists tried to extend Newton's analytical methods to the three-body problem (e.g., sun, earth, and moon) but curiously this problem turned out to be much more difficult to solve. After decades of effort, it was eventually realised that the three-body problem was essentially *impossible* to solve, in the sense of obtaining explicit formulas for the motions of the three bodies. At this point the situation seemed hopeless.

The breakthrough came with the work of Poincaré in the late 1800s. He introduced a new point of view that emphasised qualitative rather than quantitative questions. For example, instead of asking for the exact positions of the planets at all times, he asked "Is the solar system stable forever, or will some planets eventually fly off to infinity?" Poincaré developed a powerful geometric approach to analysing such questions. That approach has flowered into the modern subject of dynamics, with applications reaching far beyond celestial mechanics. Poincaré was also the first person to glimpse the possibility of chaos, in which a deterministic system exhibits aperiodic behaviour that depends sensitively on the initial conditions, thereby rendering long-term prediction impossible.

But chaos remained in the background in the first half of this century; instead dynamics was largely concerned with nonlinear oscillators and their applications in physics and engineering. Nonlinear oscillators played a vital role in the development of such technologies as radio, radar, phase-locked loops, and lasers. On the theoretical side, nonlinear oscillators also stimulated the invention of new mathematical techniques—pioneers in this area include van der Pol, Andronov, Littlewood, Cartwright, Levinson, and Smale. Meanwhile, in a separate development, Poincaré's geometric methods were being extended to yield a much deeper understanding of classical mechanics, thanks to the work of Birkhoff and later Kolmogorov, Arnold, and Moser.

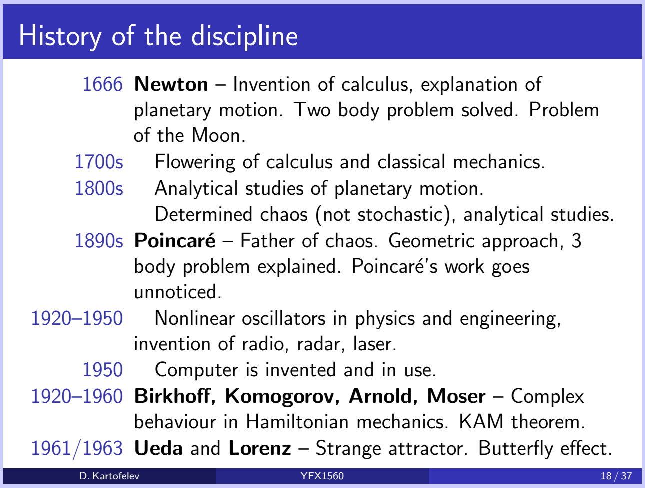
The invention of the high-speed computer in the 1950s was a watershed in the history of dynamics. The computer allowed one to experiment with equations in a way that was impossible before, and thereby to develop some intuition about nonlinear systems. Such experiments led to Lorenz's discovery in 1963 of chaotic motion on a strange attractor[1]. He studied a simplified model of convection rolls in the atmosphere to gain insight into the notorious unpredictability of the weather. Lorenz found that the solutions to his equations never settled down to equilibrium or to a periodic state—instead they continued to oscillate in an irregular, aperiodic fashion. Moreover, if he started his simulations from two slightly different initial conditions, the resulting behaviours would soon become totally different. The implication was that the system was inherently unpredictable—tiny errors in measuring the current state of the atmosphere (or any other chaotic system) would be amplified rapidly, eventually leading to embarrassing forecasts. But Lorenz also showed that there was structure in the chaos—when plotted in three dimensions, the solutions to his equations fell onto a butterfly-shaped set of points. He argued that this set had to be "an infinite complex of surfaces"—today we would regard it as an example of a fractal.

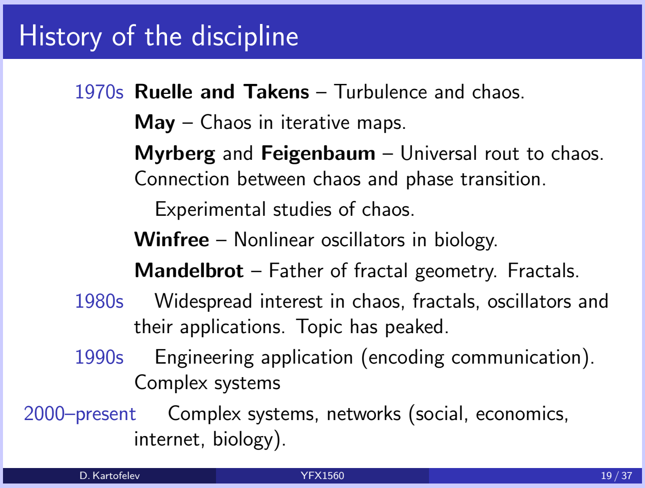
Lorenz's work had little impact until the 1970s, the boom years for chaos. Here are some of the main developments of that glorious decade. In 1971 Ruelle and Takens proposed a new theory for the onset of turbulence in fluids, based on abstract considerations about strange attractors. A few years later, May found examples of chaos in iterated mappings arising in population biology, and wrote an influential review article that stressed the pedagogical importance of studying simple nonlinear systems, to counterbalance the often misleading linear intuition fostered by traditional education. Next came the most surprising discovery of all, due to the physicist Feigenbaum[2]. He discovered that there are certain universal laws governing the transition from regular to chaotic behaviour; roughly speaking, completely different systems can go chaotic in the same way. His work established a link between chaos and phase transitions, and enticed a generation of physicists to the study of dynamics. Finally, experimentalists such as Gollub, Libchaber, Swinney, Linsay, Moon, and Westervelt tested the new ideas about chaos in experiments on fluids, chemical reactions, electronic circuits, mechanical oscillators, and semiconductors.

Although chaos stole the spotlight, there were two other major developments in dynamics in the 1970s. Mandelbrot codified and popularised fractals, produced magnificent computer graphics of them, and showed how they could be applied in a variety of subjects. And in the emerging area of mathematical biology, Winfree applied the geometric methods of dynamics to biological oscillations, especially circadian (roughly 24-hour) rhythms and heart rhythms. By the 1980s many people were working on dynamics, with contributions too numerous to list. Following slides itemise and summarise this history.

SLIDES: 18, 19

Selection of notable historic figures, their contributions to the field and the associated dates.





——— *Skip if needed:* stop ———

[1] First published works on chaotic attractors and related phenomena were authored by Japanese Electrical Engineering Professor Yoshisuke Ueda (1936–).

[2] First published works on period doubling and related phenomena were authored by Finnish mathematician Pekka Juhana Myrberg (1892–1976).

5 Introduction to theory: Basic definitions

SLIDES: 20–25

The taxonomy of mechanics presented below is generalised and simplified. Usually, in *classical* mechanics statics and dynamics are seen as a subset of rigid body mechanics.

Statics — moment about a point is zero; summation of forces is zero.

Dynamics — here moment, forces and displacement are not zero.

In this course we widen our definition of dynamics beyond the rigid body mechanics, see Slide 20.

Introduction to theory: basic definitions

Mechanics (solid and fluid) can be roughly divided into **statics** and **dynamics**.

- **Statics:** the branch of mechanics concerned with the forces acting on stationary bodies. The acting forces are in equilibrium.
- **Dynamics:** the branch of mechanics concerned with the *motion*/changes of *bodies*/systems under the action of forces. The acting forces are not in equilibrium.

The branch of any science in which changes in variables are considered e.g. chemical kinetics, population biology, nonlinear oscillations, econophysics, etc. All these subjects can be placed under a common mathematical framework.

Nonlinear dynamics: concerns with dynamical systems or processes that are inherently *nonlinear*. Nonlinear dynamical systems, describing changes in variables over time, may appear *chaotic*, *unpredictable*, or *counterintuitive*, contrasting with much simpler linear systems.

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Introduction to theory: basic definitions

In this course we will be mainly studying systems of ODEs of the form

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \tag{2}$$

where the dot $\dot{}$ denotes the time derivative (the Newton notation for differentiation), $\vec{x}$ and $\vec{f}$ are vectors and $\vec{x} \in \mathbb{R}^n$. The set $\mathbb{R}^n$ is called (n-dimensional) **phase space**.

We will also consider maps in the form

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n), \tag{3}$$

where n is the number of iterates, $\vec{x}$ is a vector and $\vec{x} \in \mathbb{R}^n$ or $\vec{x} \in \mathbb{C}^n$.

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The main notation for differentiation used in this course is the Newton's notation—so-called dot and prime notation. At times when we need to be more precise and accurate with our derivatives the Leibniz's notation ($\partial/\partial t$, $\partial/\partial x$) or the Leibniz–Newton mixed notation will be used.

Introduction to theory: basic definitions

The component form of n-th order system $\dot{\vec{x}} = \vec{f}(\vec{x})$ is

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3, \ldots, x_n), \\ \dot{x}_2 = f_2(x_1, x_2, x_3, \ldots, x_n), \\ \dot{x}_3 = f_3(x_1, x_2, x_3, \ldots, x_n), \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, x_3, \ldots, x_n). \end{cases} \tag{4}$$

Linearity: The above system is linear if function $\vec{f}$ is a linear function. Functions f_i are linear combinations of the independent variables x_i. Variables x_i appear in the first power only. No products, trigonometric, exponential, etc. functions of x_i are present.

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Introduction to theory: basic definitions

Nonlinearity: Any system that is not linear is nonlinear.

Autonomous system: no explicit dependence on time t in $\vec{f}$.

Non-autonomous system: explicitly dependant on time t in $\vec{f}$.

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3, \ldots, x_n, t), \\ \dot{x}_2 = f_2(x_1, x_2, x_3, \ldots, x_n, t), \\ \dot{x}_3 = f_3(x_1, x_2, x_3, \ldots, x_n, t), \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, x_3, \ldots, x_n, t). \end{cases} \tag{5}$$

A bulk of the time will be spent on work with the autonomous systems.

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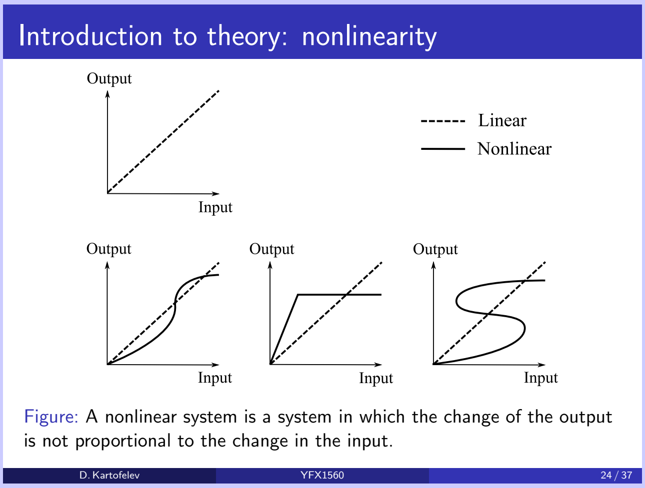
The order n of system (4) can also be understood as its dimension or more accurately the dimension of its phase space, see Slide 21.

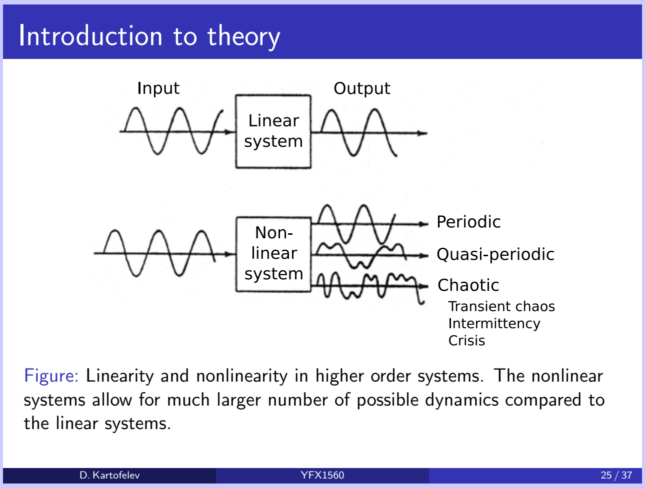
In addition to the explanation of linear functions f_i given on Slide 22, you might remember the technical definition from your linear algebra courses. A function f_i is said to be a linear if for any two vectors $\vec{x}$, $\vec{y}$ and any scalar c the following two conditions are satisfied:

$$f_i(\vec{x} + \vec{y}) = f_i(\vec{x}) + f_i(\vec{y}), \tag{1}$$

$$f_i(c\vec{x}) = c f_i(\vec{x}). \tag{2}$$

Condition (1) is called **additivity** (operation of addition) and condition (2) is called **homogeneity** (homogeneity of degree 1/operation of scalar multiplication).





NB! Discontinuities are nonlinear. Nonlinear systems may and often do produce more complex outcomes compared to linear systems.

5.1 Example: Harmonic oscillator

Are the following systems linear or nonlinear? Autonomous or non-autonomous? The equation of motion of the harmonic oscillator has the form:

$$\underbrace{m\ddot{x}}_{\substack{\text{inertial}\\ \text{term}}} + \underbrace{kx}_{\substack{\text{stiffness}\\ \text{term}}} = 0, \tag{3}$$

where x is the displacement, m is the mass and k is the stiffness (of the spring).

By introducing a variable exchange $v = \dot{x}$ (velocity) we rewrite second-order Eq. (3) as a system of first order ODEs

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dfrac{k}{m}x. \end{cases} \tag{4}$$

From here it is easy to see that this 2-D or second order system is **linear** and **autonomous**.

5.2 Example: Mathematical pendulum

The normalised dimensionless equation of motion for the mathematical pendulum has the following form:

$$\ddot{\theta} + \sin\theta = 0, \tag{5}$$

where θ is the angular displacement. By introducing a variable exchange $\omega = \dot{\theta}$ (angular velocity) we rewrite second-order Eq. (5) as a system of first order ODEs

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = -\sin\theta. \end{cases} \tag{6}$$

From here it is easy to see that this 2-D or second order system is **nonlinear** and **autonomous**.

6 1-D phase portrait

6.1 Nonlinear 1-D flow problem, analytic solution

The goal of this section is to demonstrate that some problems are much **easier to understand** when solved **using a graphical/geometric approach** introduced and explained below. For comparison we solve the following problem using the knowledge from the corses on calculus and differential equations that you may have previously taken.

SLIDES: 27–30

1-D phase portrait, 1-D flow

Example problem: 1-D flow is in the form

$$\dot{x} = \sin x, \tag{8}$$

where x is the particle position and $\dot{x}$ is the particle velocity.

Let's try to solve the problem using your usual thinking (analytical solution).

1. Separation of variables

$$\frac{dx}{dt} = \sin x \quad \Big| \cdot dt \tag{9}$$

$$dx = \sin x \, dt \quad | \div \sin x \tag{10}$$

$$\frac{dx}{\sin x} = dt \tag{11}$$

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1-D phase portrait, 1-D flow

2. Integration of the lhs and rhs

$$\int \frac{dx}{\sin x} = \int dt \tag{12}$$

lhs: $$\int \frac{dx}{\sin x} = \int \csc x \, dx = -\ln|\csc x + \cot x| + C_1 \tag{13}$$

rhs: $$\int dt = t + C_2 \tag{14}$$

Resulting equation:

$$-\ln|\csc x + \cot x| + C_1 = t + C_2 \tag{15}$$

$$-\ln|\csc x + \cot x| = t + C, \tag{16}$$

where $C = C_1 + C_2$.

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1-D phase portrait, 1-D flow

3. Resolving the integration constant C.
   Lets assume that $x = x_0$ at $t = 0$ then from (16) it follows

$$C = -\ln|\csc x + \cot x| - t \Rightarrow \begin{bmatrix} x = x_0 \\ t = 0 \end{bmatrix} \Rightarrow \tag{17}$$

$$\Rightarrow C = -\ln|\csc x_0 + \cot x_0| \tag{18}$$

4. Solution: Substitution of constant C in (16) gives

$$t = -C - \ln|\csc x + \cot x| =$$
$$= \ln|\csc x_0 + \cot x_0| - \ln|\csc x + \cot x| \tag{19}$$

$$\boxed{t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|} \tag{20}$$

Inversion of (20) in the form $t(x) = f(x, x_0)$ to $x(t) = g(t, x_0)$ is possible using trigonometric identities.

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1-D phase portrait, 1-D flow

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

How *practical* is this result?

For example can you answer the following *simple* question given that you have the analytic solution.

What is the long-term behaviour of this flow for $x_0 = \frac{\pi}{4}$?

$$\lim_{t\to\infty} x(t) \tag{21}$$

Where would you find the particle?

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6.2 Nonlinear 1-D flow problem, graphical approach

Let's answer question (21) posed on Slide 30 using the graphical approach mentioned above. This approach relies on the analysis of the system's **phase portrait**.

SLIDE: 31

1-D phase portrait, graphical approach

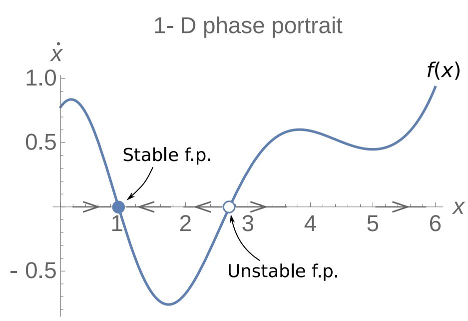

Figure: Phase portrait of the 1-D system in the form: $\dot{x} = f(x)$. Stable and unstable fixed points are shown with the filled and unfilled circles, respectively.

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Simply graph the 1-D **phase portrait** by plotting the $\dot{x}$ against x and analyse the result visually. A first-order system $\dot{x} = f(x)$ can be **regarded as a vector field on a line**—the x-axis. Think of a particle flowing on a line described by the flow direction given by sign of $\dot{x}$ and velocity $\dot{x}$.

Visual inspection and analysis of the phase portraits will prove to be a useful tool in this course. The **fixed points** shown on Slide 31 are important characteristic points corresponding to the lack of motion, i.e.,

$$\dot{x} = f(x^*) = 0, \tag{7}$$

where x^* is the coordinate of the fixed point. There are two distinct types of fixed points shown on Slide 31. The **stable fixed points** attract the flow and the **unstable fixed points** repel it. On the graphs fixed points are usually denoted by filled and/or empty bullets (• – stable fixed point, ∘ – unstable fixed point).

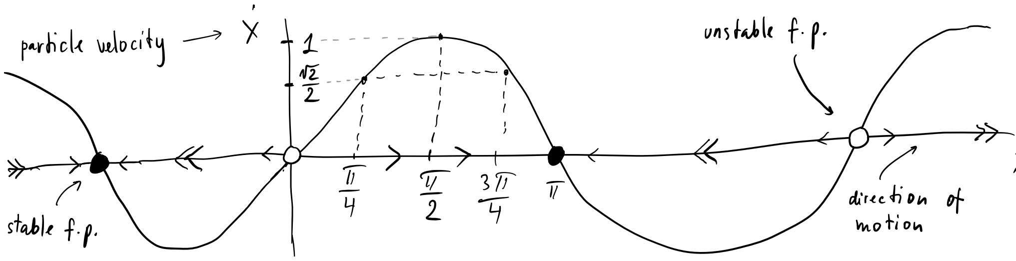


Figure 3: Phase portrait, the graph of particle velocity $\dot{x}$ vs. position x, of Eq. (8). The particle flow speed and direction are indicated with the arrowheads. Here, $>$ indicates the flow to the right and $\gg$ even faster flow to the right.

1-D phase portrait of the aforementioned problem (Slide 27) that has the form:

$$\dot{x} = \sin x, \tag{8}$$

is shown in Fig. 3. Fixed points x^* are the following

$$\dot{x} = 0 \;\Rightarrow\; \sin x^* = 0 \;\Rightarrow\; x^* = n\pi, \quad \text{where } n \in \mathbb{Z}. \tag{9}$$

A visual inspection of the phase portrait reveals the answer to the original question (Slide 30). The long-term behaviour of a particle starting at $x_0 = x(0) = \pi/4$ is

$$\lim_{t\to\infty} x(t) = \pi. \tag{10}$$

Moreover, the 1-D phase portrait shown in Fig. 3 can be used to deduce qualitatively the integrated solution $x(t)$. This method of deduction of the solution is shown in Fig. 4.

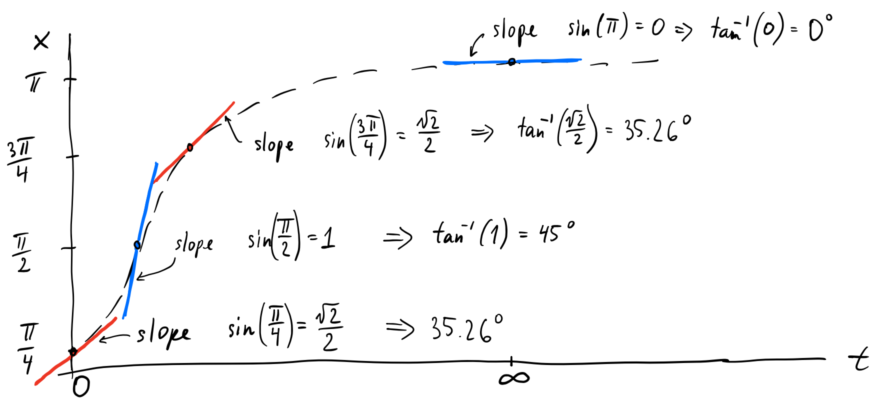


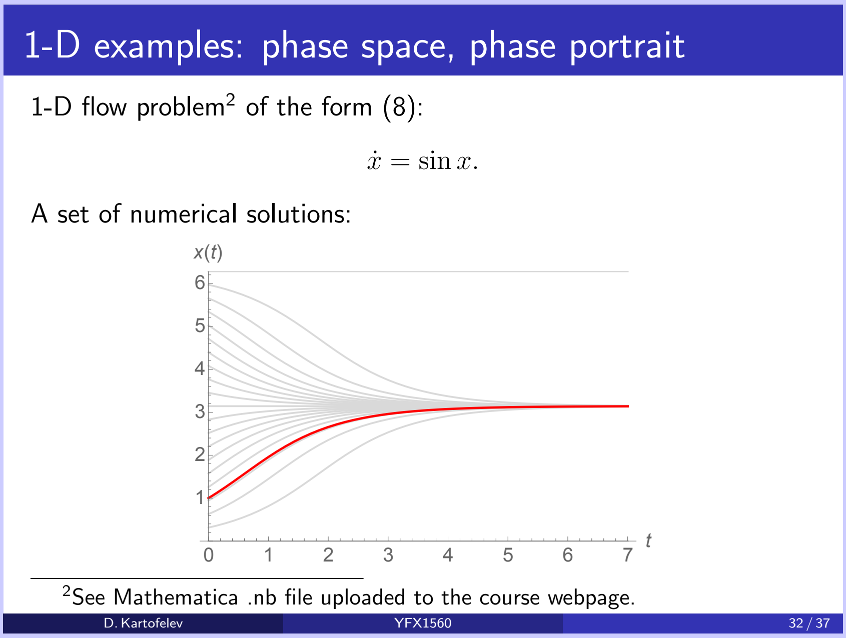
Figure 4: Qualitative solution of Eq. (8) for $x_0 = x(0) = \pi/4$.

Let's check the obtained result against a numerically integrated solution.

NUMERICS: NB#2

Analytic solution to a nonlinear ODE. Numerical solution and phase portrait of 1-D flow $\dot{x} = \sin x$.

SLIDE: 32



6.3 The logistic equation, graphical approach

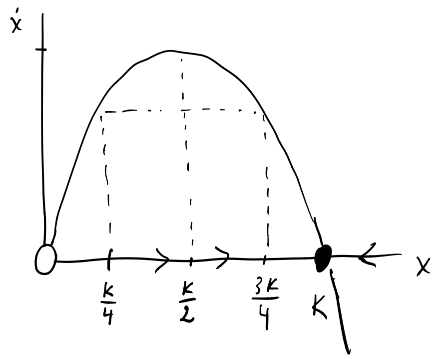


Figure 5: Phase portrait of the logistic equation given by Eq. (11).

The logistic equation is an example of a 1-D nonlinear problem from ecology—a simplified model of population dynamics in isolation. The model has the following form:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \tag{11}$$

where $x \geq 0$ is the size of the population, r is the reproduction rate, K is the carrying capacity of the finite ecology. Also, we assume that $\{r, K\} \in \mathbb{R}$ and $r, K > 0$.

Non-negative fixed points x^* are the roots of the following quadratic polynomial

$$\dot{x} = 0 \;\Rightarrow\; rx^*\left(1 - \frac{x^*}{K}\right) = 0 \;\Rightarrow\; \begin{cases} x_1^* = 0, \\ x_2^* = K. \end{cases} \tag{12}$$

Figure 5 shows the phase portrait and the corresponding fixed points. All the information regarding the dynamics is clearly visible. For example the long-term behaviour for $x_0 = x(0) > 0$ is always K,

$$\lim_{t\to\infty} x(t) = K. \tag{13}$$

In biology it is beneficial to consider the population growth rate per capita

$$\frac{\dot{x}}{x} = r\left(1 - \frac{x}{K}\right). \tag{14}$$

The $\dot{x}/x$ vs. x graph is shown in Fig. 6. The graph shows clearly that if $x < K$, then $\dot{x}/x > 0$, i.e., the population is growing and if $x > K$, than $\dot{x}/x < 0$, i.e., the population is decreasing.

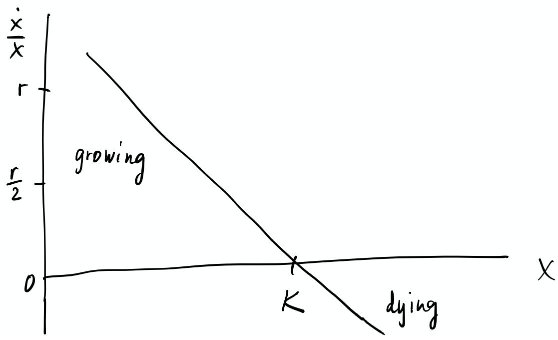


Figure 6: Growth rate per capita, Eq. (14) where K is the carrying capasity.

The numerical solution of the logistic equation is linked and presented below.

NUMERICS: NB#3

Numerical solution and phase portrait of the 1-D logistic equation.

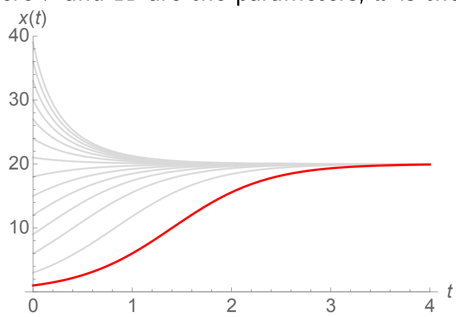
SLIDE: 33

1-D examples: phase space, phase portrait

The logistic equation[3] (population dynamics in isolation):

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \tag{22}$$

where r and K are the parameters, x is the size of the population.



What is the value of carrying capacity K here?

[3]See Mathematica .nb file uploaded to the course webpage.

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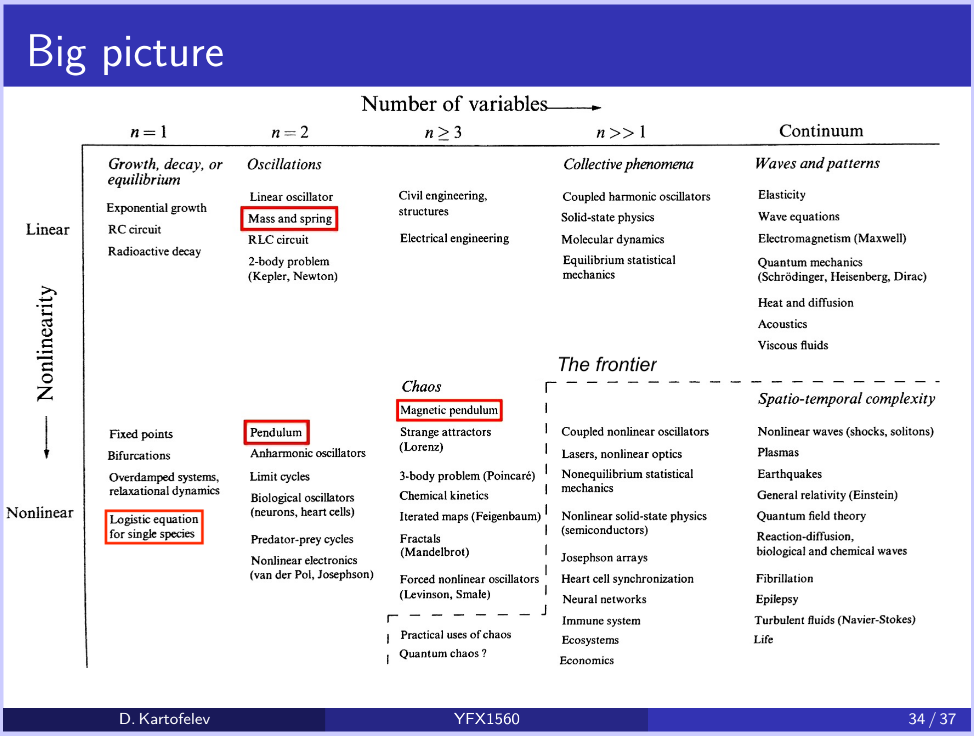
7 Conclusions and general introduction

Now that we have established the ideas of **linearity**, **nonlinearity** and **phase space**, we can present a framework for dynamics and its applications. Here, our goal is to show the bigger picture of the entire subject. The framework presented on Slide 34 will guide our studies throughout this course. The framework has two axes. **One axis** tells us the number of variables needed to characterise the state of the system. Equivalently, this number is the *dimension of the phase space*. The **other axis** tells us whether the system is linear or nonlinear.

Upper part of the graphs is more familiar to you because of your previous studies. In contrast, the lower half of the graph—the nonlinear half—is often ignored or deferred to later courses. But no more! In this course **we start in the lower left corner and systematically head to the right**. As we increase the phase space dimension from $n = 1$ to $n = 3$, we encounter **new phenomena at every step**, from fixed points and bifurcations when $n = 1$, to nonlinear oscillations when $n = 2$, and finally chaos and fractals when $n = 3$. In all cases, a **geometric approach proves to be very powerful**, and gives us most of the information we want, even though we usually can't solve the equations in the traditional sense of finding a formula for the answer.

SLIDE: 34



The problems that were mentioned/discussed during this lecture are highlighted with red rectangles.

Revision questions

1. What is dynamics?
2. Name a dynamical system.
3. Define nonlinearity.
4. Determine if the following equations/systems are linear or nonlinear:

$$\dot{x} = \sin x, \tag{15}$$

$$\dot{x} = \ln x, \tag{16}$$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = xy, \end{cases} \tag{17}$$

$$\ddot{x} + \dot{x} + x = 0. \tag{18}$$

5. What is ordinary homogeneous differential equation?
6. Define 1-D dynamical system. Name a 1-D problem.
7. What is phase space?
8. What is phase portrait?
9. Sketch 1-D phase portrait of the following systems:

$$\dot{x} = 3\cos x, \tag{19}$$

$$\dot{x} = 0.5x^2 - 1, \tag{20}$$

$$\dot{x} = x^3. \tag{21}$$

10. What is a fixed point?
11. How to find a fixed point of a differential equations?
12. Find the fixed point or points of the following system:

$$\ddot{x} + \dot{x} + x = 0. \tag{22}$$

13. Explain fixed point stability.
14. What is linear analysis of a system?