Lecture №9: Attractor and strange attractor, chaos, analysis of the Lorenz attractor, the Lyapunov exponents, predictability horizon, examples of chaos, 1-D map

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Lecture outline

- Analysis and properties of the Lorenz attractor dynamics
- The Lyapunov exponents and Kolmogorov entropy
- The Lyapunov time or the predictability horizon
- Conceptual definition of chaos or deterministic chaos
- Conceptual definition of attractor and strange attractor
- Examples of chaotic systems and attractors
  - Dynamics of the Solar System
- 1-D maps, cobweb diagram and recurrence map (relation)
- Examples discussed:
  - Various strange attractors
  - The double mathematical pendulum
  - The gravitational three-body problem
  - Magnetic pendulum in three magnetic potentials
  - The Lorenz map
- Differences between chaotic behaviours

The Lorenz attractor

The **Lorenz attractor** has the form

$$\begin{cases} \dot{x} = d\,(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where r, d, and b are the parameters. For $r = 28$, $d = 10$, and $b = 8/3$, the system has a chaotic solution. Other parameter values may generate other types of solutions.

The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions and forward osmosis.

Properties of the Lorenz attractor

- There exists a symmetric pair of solutions. If $(x, y) \to (-x, -y)$ the system stays the same. If solution $(x(t), y(t), z(t))$ exists then solution $(-x(t), -y(t), z(t))$ is also a solution.
- The Lorenz system is **dissipative**[1]: volumes V in phase space contract under the flow. As $t \to \infty$, $V \to 0$.

$$\dot{V} = \int_V \nabla \cdot \dot{\vec{x}} \, \mathrm{d}V \tag{2}$$

$$\nabla \cdot \dot{\vec{x}} = \frac{\partial}{\partial x}[d(y - x)] + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) = \\ = -(d + 1 + b) < 0 = \text{const.} \tag{3}$$

Since the divergence is constant, (2) reduces to

$$\dot{V} = -(d + 1 + b)\, V \quad \Rightarrow \quad V(t) = V(0)\mathrm{e}^{-(d+1+b)t}. \tag{4}$$

[1]See Mathematica .nb file uploaded to the course webpage.

Bifurcation analysis of the Lorenz attractor

For $r < 1$ ($d = 10$, $b = 8/3$) there is only one stable fixed point located at the origin. This point corresponds to no convection. All orbits converge to the origin — a global attractor.
A **supercritical pitchfork bifurcation** occurs at $r = 1$, and for $r > 1$ two additional fixed points appear at

$$(x^*, y^*, z^*) = C^{\pm} = \left(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1 \right). \qquad (5)$$

These correspond to steady convection. Fixed points are stable only if

$$r < r_{\text{Hopf}}, \quad r_{\text{Hopf}} = d\,\frac{d+b+3}{d-b-1} = 24.74, \qquad (6)$$

which can hold only for positive r and $d > b + 1$. At a critical value $r = r_{\text{Hopf}}$, both stable fixed points lose stability through a **subcritical Hopf** bifurcation.

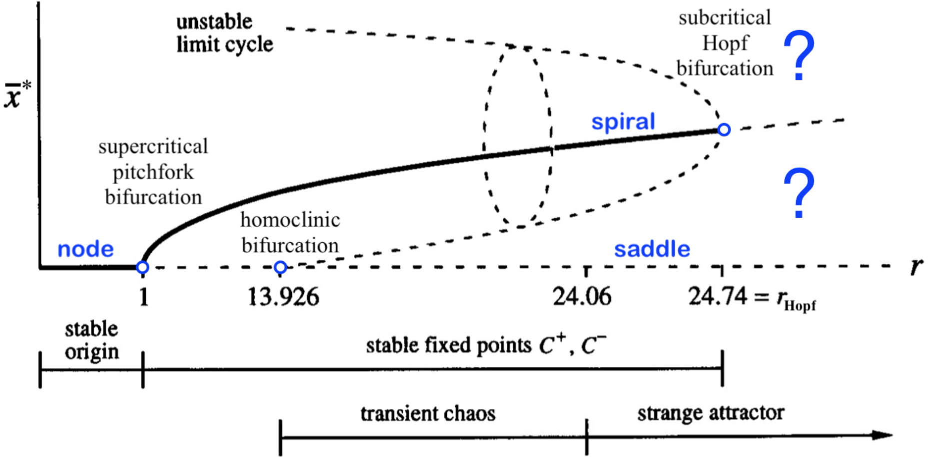
Bifurcation analysis of the Lorenz attractor

As we decrease r from r_{Hopf}, the unstable limit cycles expand and pass precariously close to the **saddle point** at the origin. At $r = 13.926$ the cycles touch the saddle point and **become homoclinic orbits**; hence we have a **homoclinic bifurcation** which is referred to as the *first homoclinic explosion*. Below $r = 13.926$ there are no limit cycles.

The region $13.926 < r < 24.06$ is referred to as **transient chaos**[2] region. Here, the chaotic trajectories eventually settle at C^+ or C^-.

[2] See Mathematica .nb file uploaded to the course webpage.

Bifurcation analysis of the Lorenz attractor



Bifurcation analysis of the Lorenz attractor

For $r > 24.06$ and $r > r_{\text{Hopf}} = 28$ (immediate vicinity): no stable limit-cycles exist; trajectories do not escape to infinity (dissipation); do not approach an invariant torus (quasi-periodicity). Almost all initial conditions (I.C.s) will tend to an invariant set — the Lorenz attractor — a **strange attractor** and a **fractal**.

Note: No quasi-periodic solutions for $r > r_{\text{Hopf}}$ are possible because of the dissipative property of the flow.

For $r \gg r_{\text{Hopf}}$ different types of chaotic dynamics exist, e.g. **noisy periodicity, transient and intermittent chaos**. One can even find transient chaos settling to periodic orbits[3].

[3] See Mathematica .nb file uploaded to the course webpage.

The Lyapunov exponent and predictability horizon

Example: Suppose we're trying to predict the future state of a chaotic system within a tolerance of $a = 10^{-3}$. Given that our estimate of the initial state is uncertain to within $\delta_0 = 10^{-7}$, for about how long can we predict the state of the system, while remaining within the tolerance?

Now suppose we manage to measure the initial state a *million* times better, i.e., we improve our initial error to $\delta_0 = 10^{-13}$. How much longer can we predict?

The Lyapunov exponent and predictability horizon

Solution: The original prediction has

$$t = \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-7}} = \frac{1}{\lambda} \ln(10^4) = \frac{4 \ln 10}{\lambda}. \tag{7}$$

The improved prediction has

$$t = \frac{1}{\lambda} \ln \frac{10^{-3}}{10^{-13}} = \frac{1}{\lambda} \ln(10^{10}) = \frac{10 \ln 10}{\lambda}. \tag{8}$$

Thus, after a millionfold improvement in our initial uncertainty, we can predict only $10/4 = 2.5$ times longer (system's timeframe)!

Conclusions: If one wants to predict further into the future the task becomes exponentially *harder*. Since,

$$t \simeq \frac{1}{\lambda} \ln \frac{a}{\delta_0} \sim \ln \frac{a}{\delta_0} \quad \Rightarrow \quad \frac{a}{\delta_0} \sim \mathrm{e}^t. \tag{9}$$

Conceptual definitions

Chaos a long-term aperiodic behaviour in a *deterministic* system that exhibits sensitive dependents on I.C.s ,i.e., system has positive Lyapunov exponent λ.

Intermittency or intermittent chaos is the irregular alternation of phases of apparently periodic and chaotic dynamics, or different forms of chaotic dynamics (*crisis*-induced intermittency).

Transient chaos is temporary short-lived chaos that is replaced gradually or abruptly by periodic dynamics or stable fixed point/s.

Crisis is the sudden appearance or disappearance of a *strange attractor* as the parameters of a dynamical system are varied.

Conceptual definitions

Attractor meets the following criteria. Set A is:
1) Invariant set (start in A and stays in A for $t \to \infty$).
2) Attracts open set U of I.C.s. U is basin of attraction.
3) Is minimal (smallest set). There are no proper sub-sets of A that satisfy (1) and (2).

Strange attractor an attractor that exhibits *sensitive* dependents on I.C.s: The Lyapunov exponent $\lambda > 0$. Local geometric structure (manifold) is *fractal*.

Chaotic attractor when emphasising the chaotic property of an attractor.

Fractal attractor when emphasising the fractal geometry of an attractor.

Conceptual definitions

Strange non-chaotic attractor (SNA) an attractor with long-term non-chaotic aperiodic dynamics. Such attractors are generic in quasi-periodically driven nonlinear systems, and like strange attractors, are geometrically *fractal*. The largest Lyapunov exponent $\lambda \leq 0$: trajectories do not show exponential sensitivity to I.C.s.

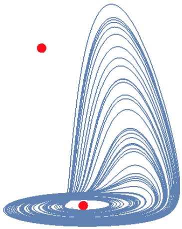
Note: A system can be chaotic but not an attractor.
Chaotic attractors and other types of dynamics can co-exist in a single system.

The Rössler attractor

The **Rössler attractor**[4] has the form

$$\begin{cases} \dot{x} = -y - x, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases} \tag{10}$$

Chaotic solution exists for $a = 0.1$, $b = 0.1$, $c = 14$.



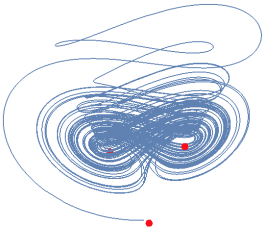
[4]See Mathematica .nb file uploaded to the course webpage.

The Chen attractor

The **Chen attractor**[5] has the form

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz. \end{cases} \tag{11}$$

Chaotic solution exists for $a = 35$, $b = 3$, $c = 28$.



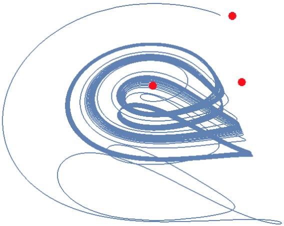
[5]See Mathematica .nb file uploaded to the course webpage.

The modified Chen attractor

The **modified Chen attractor**[6] has the form

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy + m, \\ \dot{z} = xy - bz. \end{cases} \tag{12}$$

Chaotic solution exists for $a = 35$, $b = 3$, $c = 28$, $m = 23.1$.



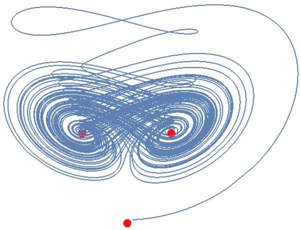
[6]See Mathematica .nb file uploaded to the course webpage.

The Lü attractor

The **Lü attractor**[7] has the form

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -xz + cy, \\ \dot{z} = xy - bz. \end{cases} \tag{13}$$

Chaotic solution exists for $a = 36$, $b = 3$, $12.7 < c < 17.0$ (similar to the Lorenz system), $23.0 < c < 28.5$ (similar to the Chen system).



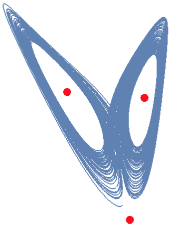
[7]See Mathematica .nb file uploaded to the course webpage.

The Pan-Xu-Zhou attractor

The **Pan-Xu-Zhou attractor**[8] has the form

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - xz, \\ \dot{z} = xy - bz. \end{cases} \tag{14}$$

Chaotic solution exists for $a = 10$, $b = 8/3$, $c = 16$.



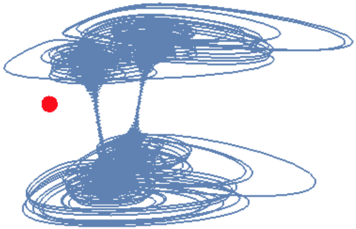
[8]See Mathematica .nb file uploaded to the course webpage.

The Bouali attractor

The **Bouali attractor**[9] has the form

$$\begin{cases} \dot{x} = \alpha(1-x)x - \beta z, \\ \dot{y} = -\gamma(1-x^2)y, \\ \dot{z} = \mu x. \end{cases} \tag{15}$$

Chaotic solution exists for example for $\alpha = 3.0$, $\beta = 2.2$, $\gamma = 1.0$ and $\mu = 0.001$.



[9] See Mathematica .nb file uploaded to the course webpage.

Strange attractors

Some other strange attractors:

- The Lorenz 84 attractor
- The Newton–Leipnik chaotic system
- The Thomas' cyclically symmetric attractor
- The Sprott attractors (Sprott A – S)
- The algebraically simplest dissipative chaotic flow (based on jerk equation derived by J. S. Sprott)

$$\dddot{x} + A\ddot{x} \pm \dot{x}^2 + x = 0, \qquad (16)$$

where A is constant taken as 2.017, for example.

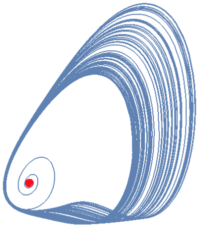


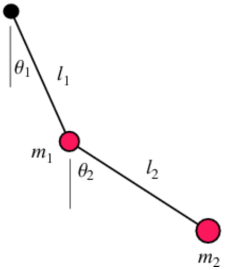
Figure: The algebraically simplest dissipative chaotic flow.

The double mathematical pendulum

The **double pendulum:** System is implicit for $l_1 \neq l_2$.

$$\begin{cases} (m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) \\ \qquad\qquad + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + g(m_1+m_2)\sin\theta_1 = 0 \\ m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) \\ \qquad\qquad - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 g \sin\theta_2 = 0, \end{cases} \tag{17}$$

where



See Mathematica .nb file uploaded to the course webpage.

The double mathematical pendulum

No embedded video files in this pdf

Magnetic pendulum in three magnetic potentials

System is modeled with the following equations of motion:

$$\begin{cases} \ddot{x} + R\dot{x} - \displaystyle\sum_{i=1}^{N} \frac{x_i - x}{\left(\sqrt{(x_i - x)^2 + (y_i - y)^2 + d^2}\right)^3} + Cx = 0, \\ \ddot{y} + R\dot{y} - \displaystyle\sum_{i=1}^{N} \frac{y_i - y}{\left(\sqrt{(x_i - x)^2 + (y_i - y)^2 + d^2}\right)^3} + Cy = 0, \end{cases} \tag{18}$$

where R is proportional to the air resistance and overall attenuation, C is proportional to the effects of gravity, N is the number of magnets, the i-th magnet is positioned at (x_i, y_i), d is the distance between the pendulum at rest and the plane of magnets.

Additionally we assume that the pendulum length is long compared to the spacing of the magnets. Thus, we may assume for simplicity that the metal ball moves about on a xy-plane.

Magnetic pendulum in three magnetic potentials[10]

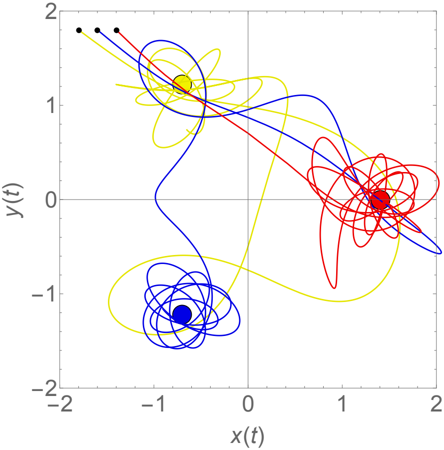


Figure: Magnets shown with the yellow, red and blue colours, attracting the magnetic pendulum for three neighbouring initial conditions.

[10] See Mathematica .nb file uploaded to the course webpage (Lecture №1).

Magnetic pendulum

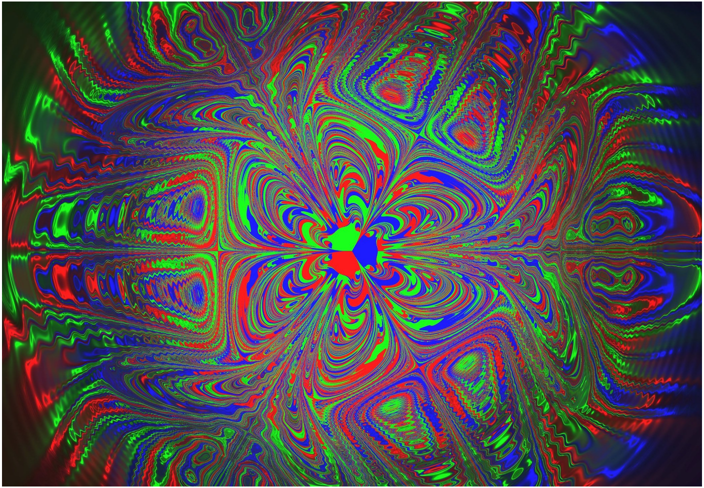


Figure: The **basins of attraction** of the three magnets which are coloured red, blue and green.

D. Berger, https://twitter.com/InertialObservr

Magnetic pendulum

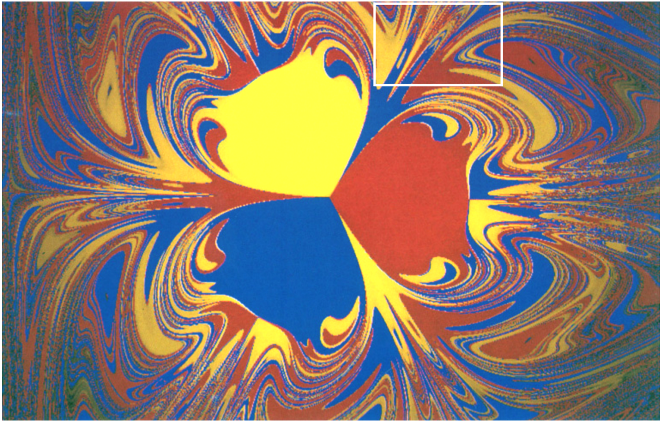


Figure: The **basins of attraction** of the three magnets which are coloured red, blue and yellow.

H. Peitgen, *et al*, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, 2004, pp. 707–714.

Magnetic pendulum

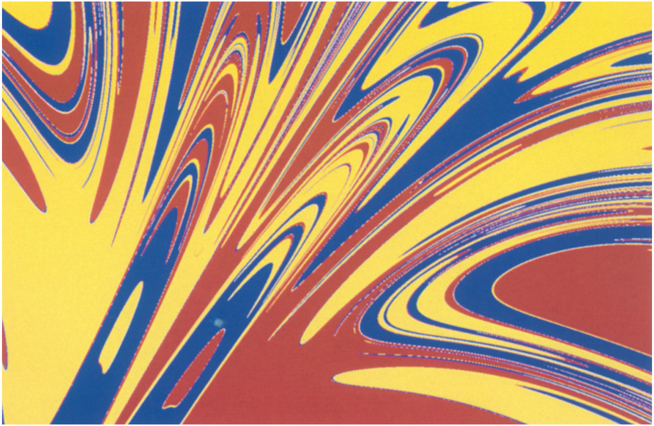


Figure: Detail of previous slide showing the intertwined structure of the three basins.

H. Peitgen, *et al*, *Chaos and Fractals: New Frontiers of Science*, Springer-Verlag, 2004, pp. 707–714.

The gravitational three-body problem

The **gravitational three-body problem**[11]: system consists of 18 first order equations.

$$\ddot{\vec{r}}_i = \sum_{j=1,\, j\neq i}^{3} -Gm_j \, \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}, \quad i = 1, 2, 3, \tag{19}$$

where $\vec{r}_i = (x_i, y_i, z_i)$ is the ith body's position vector, $\ddot{\vec{r}}_i$ is the acceleration of ith body, m_j is the jth mass, $\vec{r}_i - \vec{r}_j$ is the vector connecting the masses i and j, G is the gravitational constant, and $|\vec{\alpha}|$ denotes the vector norm of vector $\vec{\alpha}$.

[11] See Mathematica .nb file uploaded to the course webpage. Numerical solution of the planar problem where $\vec{r}_i = (x_i, y_i)$ is uploaded.

The gravitational three-body problem[12]

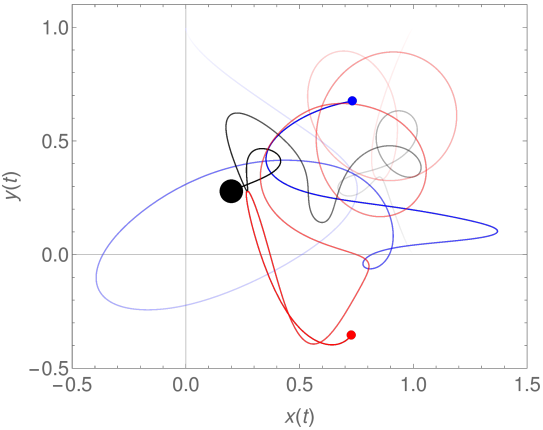


Figure: Numerical solution of the planar three-body problem.

[12] See Mathematica .nb file uploaded to the course webpage. Numerical solution of the planar problem where $\vec{r}_i = (x_i, y_i)$ is uploaded.

Dynamics of the Solar System

Is our solar system chaotic or not?

Read: J. Laskar, "Large-scale chaos in the solar system," *Astronomy and Astrophysics*, **287**(1), pp. L9–L12, (1994).

Consider also other works by J. Laskar.

Read: Wayne B. Hayes, Anton V. Malykh, Christopher M. Danforth, "The interplay of chaos between the terrestrial and giant planets," *Monthly Notices of the Royal Astronomical Society*, **407**(3), pp. 1859–1865, (2010).

Introduction to 1-D maps

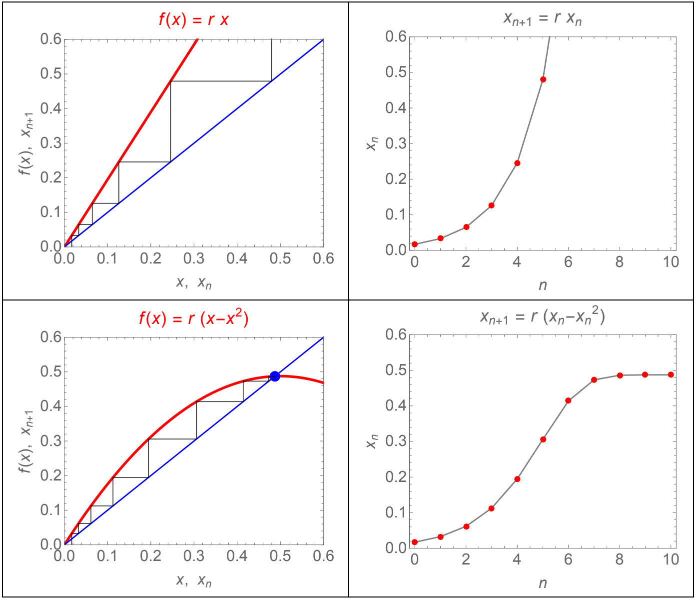
The general form of 1-D map is the following:

$$x_{n+1} = f(x_n), \tag{20}$$

where f is the given function, $n \in \mathbb{Z}^+$ is the number of iterates applied to an initial condition $x_0 \in \mathbb{R}$ or $x_0 \in \mathbb{C}$.

A graphical way of generating the iterates x_n is to construct a **cobweb diagram**.

1-D maps: Cobweb diagram and map iterates



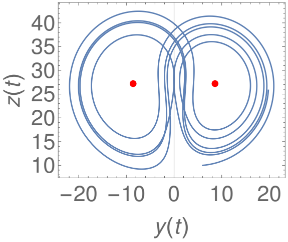
Chaotic dynamics in the Lorenz attractor

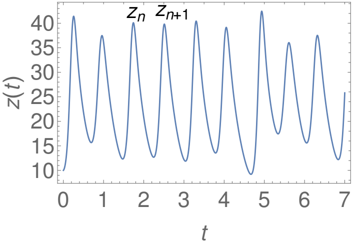
How do we know that the Lorenz attractor is not just a stable limit-cycle in disguise?

Playing devil's advocate, a skeptic might say, "Sure, the trajectories don't ever seem to repeat, but maybe you haven't integrated long enough. Eventually the trajectories will settle down into a periodic behaviour — it just happens that the **period is incredibly long**, much longer than you've tried in your computer. Prove me wrong."

Chaotic dynamics in the Lorenz attractor

Lorenz directs our attention to a particular view of the attractor.





Lorenz writes: "the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again. It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit." (Lorenz 1963)

The feature that he focuses on is z_n the n-th local maximum of $z(t)$.

Chaotic dynamics in the Lorenz attractor

Lorenz's idea is that z_n should predict z_{n+1}. To check this, he numerically integrated the equations for $t \gg 1$, then found the local maxima of $z(t)$, and finally plotted z_{n+1} vs. z_n. The data from the chaotic time series appear to fall neatly on a curve.

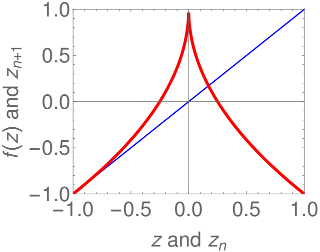


Figure: The Lorenz map shown with red where $|f'(z)| > 1$ for $\forall z \neq 0$.

By this ingenious trick, Lorenz was able to extract order from chaos. The normalised map $z_{n+1} = f(z_n)$ shown above is now called the **Lorenz map**.

Ruling out stable limit-cycles

Stability of fixed point z^* can be determined by linearisation, consider a slightly perturbed trajectory close to fixed point z^*

$$z_n = z^* + \eta_n, \tag{21}$$

where $|\eta| \ll 1$. Linearisation of the map at z^* yields

$$z_{n+1} = f(z_n) = f(z^* + \eta_n) = \begin{bmatrix} \text{Taylor series} \\ \text{about } z_n = z^* \end{bmatrix} = \\ = f(z^*) + \frac{f'(z^*)}{1!}(z^* + \eta_n - z^*) + O(\eta_n^2) \approx z^* + f'(z^*)\eta_n. \tag{22}$$

$$\underbrace{z_{n+1} - z^*}_{\eta_{n+1}} \approx f'(z^*)\eta_n \tag{23}$$

Ruling out stable limit-cycles

Linearisation of the map at z^* yields

$$\boxed{\eta_{n+1} \approx |f'(z^*)|\eta_n} \tag{24}$$

Since $|f'(z^*)| > 1$, by the Lorenz map property, we get

$$|\eta_{n+1}| > |\eta_n|. \tag{25}$$

Hence, the deviation η_n grows with each iteration. Fixed point z^* is unstable, and all orbits must be unstable.
General conclusions from linearisation:

$$|f'(z^*)| < 1, \quad z^* \text{ is stable.} \tag{26}$$

$$|f'(z^*)| = 1, \quad z^* \text{ participates in bifurcation.} \tag{27}$$

$$|f'(z^*)| > 1, \quad z^* \text{ is unstable.} \tag{28}$$

Conclusions

- Analysis and properties of the Lorenz attractor dynamics
- The Lyapunov exponents and Kolmogorov entropy
- The Lyapunov time or the predictability horizon
- Conceptual definition of chaos or deterministic chaos
- Conceptual definition of attractor and strange attractor
- Examples of chaotic systems and attractors
    - Dynamics of the Solar System
- 1-D maps, cobweb diagram and recurrence map (relation)
- Examples discussed:
    - Various strange attractors
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    - The Lorenz map
- Differences between chaotic behaviours

Revision questions

- Define attractor.
- Define strange attractor.
- What is the difference between a strange attractor and an attractor?
- Name properties of the Lorenz attractor.
- What are the Lyapunov exponents?
- What is the Lyapunov exponent?
- What determines the number of Lyapunov exponents?
- What is the Kolmogorov entropy?
- What is predictability horizon?
- What is the Lyapunov time?
- Can a long-term solution to a chaotic system be predicted? Explain.
- List some examples of chaos in nature.

Revision questions

- What is final-state sensitivity?
- What is chaos?
- What is intermittent chaos?
- What is transient chaos?
- What is crisis?
- What is strange non-chaotic attractor?