Lecture №7: Bifurcations in 2-D, bifurcations of fixed points, the Hopf bifurcation, bifurcations of closed orbits, examples of dynamical instabilities

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TAL TECH
SCHOOL OF SCIENCE

Lecture outline

- Classification of bifurcations in 2-D systems
- Bifurcations of fixed points
- Bifurcations of closed orbits
- The Hopf bifurcation
- Hysteresis on the level of cycles
- Videos of examples of dynamical instabilities (engineering, chemistry, neurology)

Classification of bifurcations in 2-D

Case I Bifurcations of fixed points

A) Bifurcations at $\lambda = 0$

1) Saddle-node bifurcation
2) Transcritical bifurcation
3) Pitchfork bifurcation
   - Supercritical pitchfork bifurcation
   - Subcritical pitchfork bifurcation

B) Hopf bifurcations, bifurcations at $\lambda = \pm i\omega$

1) Supercritical Hopf bifurcation
2) Subcritical Hopf bifurcation

Case II Bifurcations of closed orbits

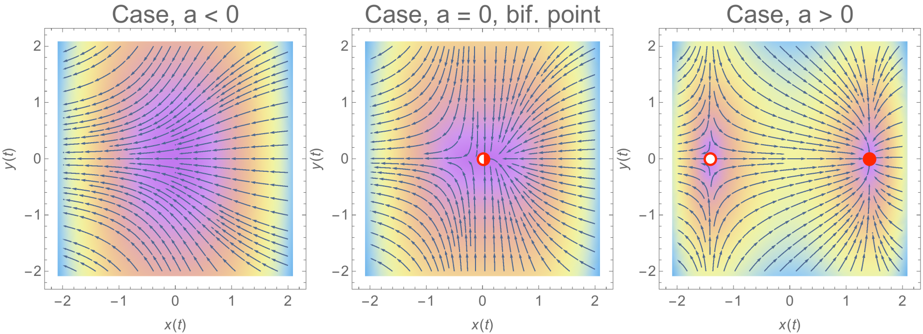
A) Saddle-node coalescence of cycles (accompanied by subcritical Hopf)

B) SNIPER (saddle-node infinite period bif.) or SNIC (saddle-node in invariant cycle bif.)

C) Homoclinic bifurcation or saddle-loop bifurcation

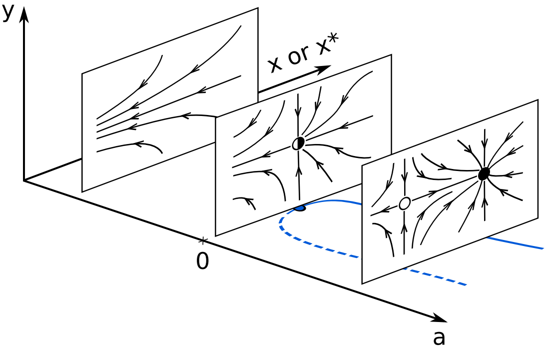
Case I A 1: Saddle-node bifurcation.
Normal form:

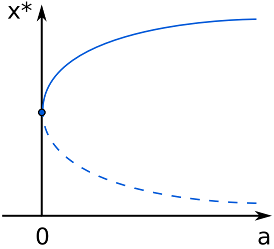
$$\begin{aligned} \dot{x} &= a - x^2 \\ \dot{y} &= -y \end{aligned} \tag{1}$$



Case I A 1: Saddle-node bifurcation.
Normal form:

$$\dot{x} = a - x^2$$
$$\dot{y} = -y$$

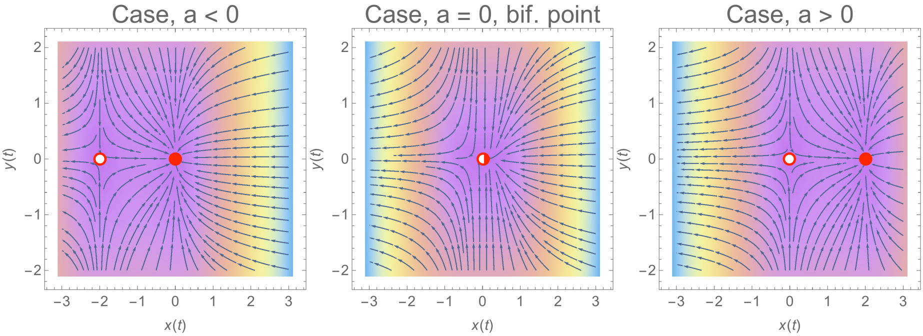




Case I: Bifurcations of fixed points

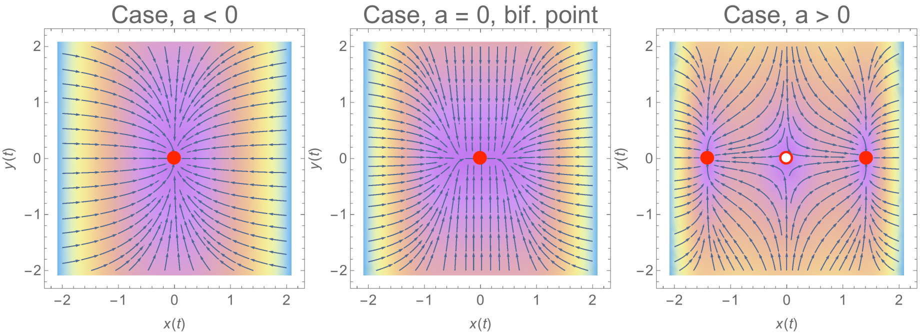
Case I A 2: Transcritical bifurcation.
Normal form:

$$\begin{aligned} \dot{x} &= ax - x^2 \\ \dot{y} &= -y \end{aligned} \tag{2}$$



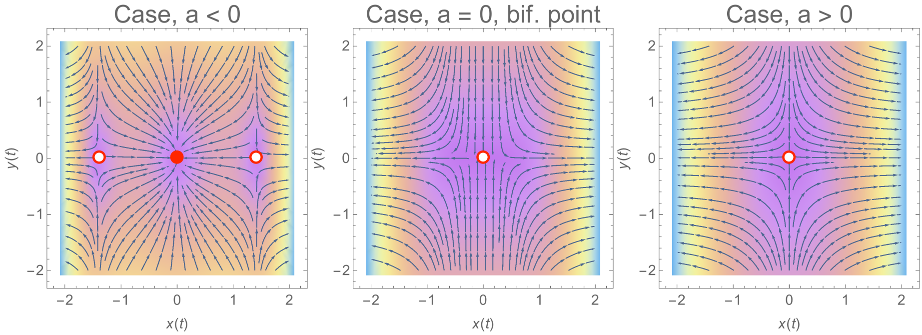
Case I A 3: Supercritical pitchfork bifurcation.
Normal form:

$$\begin{aligned}\dot{x} &= ax - x^3 \\ \dot{y} &= -y\end{aligned} \tag{3}$$



Case I A 3: Subcritical pitchfork bifurcation.
Normal form:

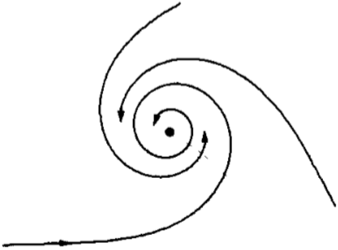
$$\dot{x} = ax + x^3$$
$$\dot{y} = -y \tag{4}$$



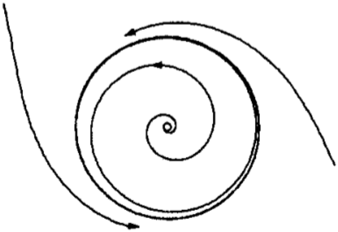
Case I: Bifurcations of fixed points

Case I B 1: The supercritical Hopf bifurcation. Bif. parameter is μ.
Normal form:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases} \tag{5}$$



$\mu < 0$

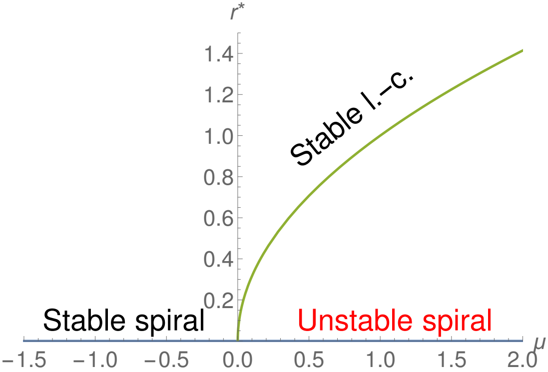


$\mu > 0$

Case I: Bifurcations of fixed points

Case I B 1: The supercritical Hopf bifurcation. Bif. parameter is μ.
Normal form:

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Case I: Bifurcations of fixed points

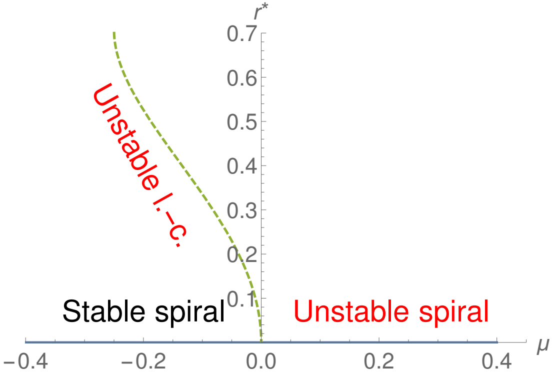
Case I B 2: The subcritical Hopf bifurcation. Bif. parameter is μ. Normal form:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \tag{6}$$

Case I: Bifurcations of fixed points

Case I B 2: The subcritical Hopf bifurcation. Bif. parameter is μ. Normal form:

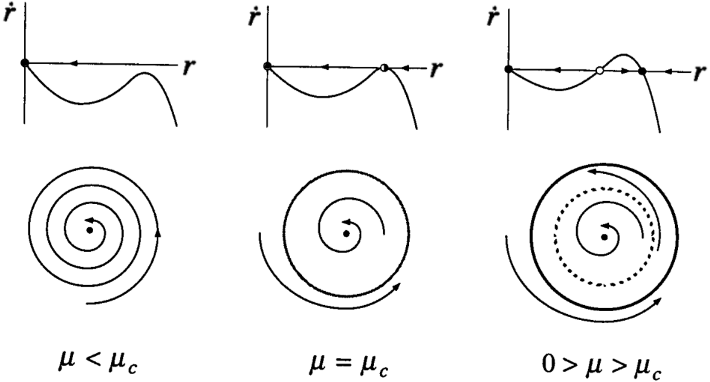
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$



Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

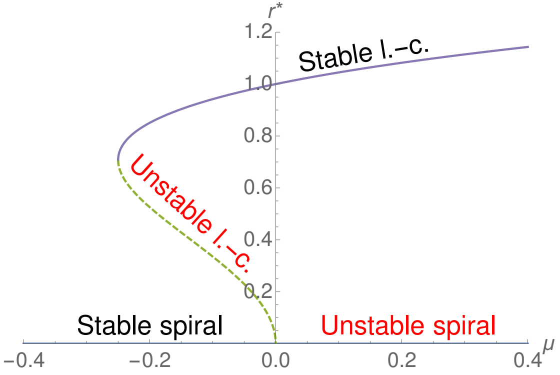
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases} \tag{7}$$



Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

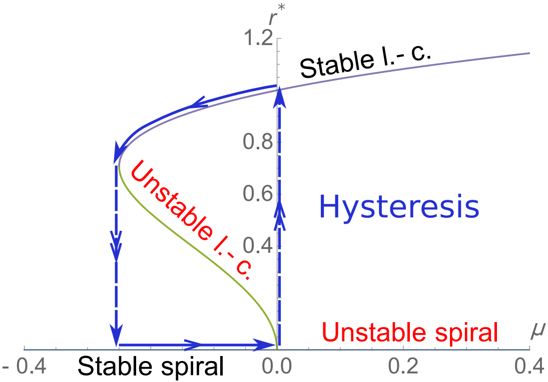
$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Case II: Bifurcations of closed orbits

Case II A: Saddle-node coalescence of cycles. Bif. parameter is μ and $\mu_c = -1/4$. Example system:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + br^2 \end{cases}$$



Aeroelastic flutter

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The Tacoma Narrows bridge collapse, 1940

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The Briggs–Rauscher oscillating reaction

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The Belousov–Zhabotinsky reaction

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The tremor-dominant Parkinson's disease

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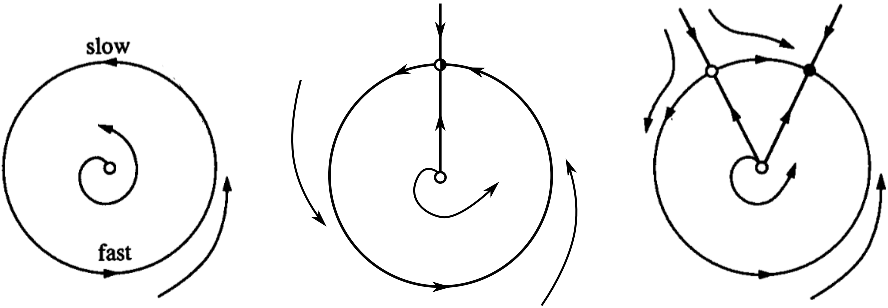
Case II B: SNIPER or SNIC. Bif. parameter is ω.
Example system:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \omega - \sin\theta \tag{8}$$

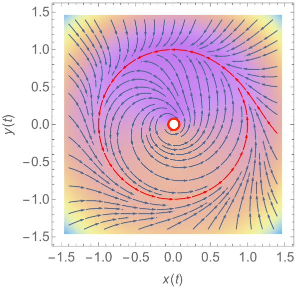


$\omega > 1$ $\omega = 1$ $\omega < 1$

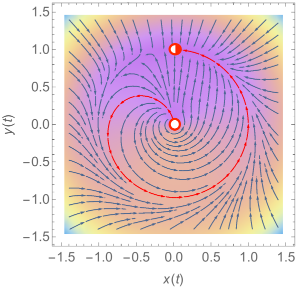
Case II: Bifurcations of closed orbits

Case II B: SNIPER or SNIC. Bif. parameter is ω.
Example system:

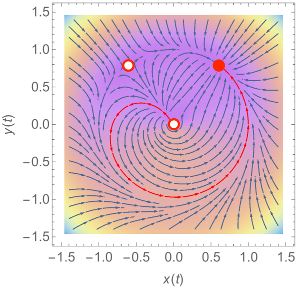
$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \omega - \sin\theta$$



$\omega > 1$



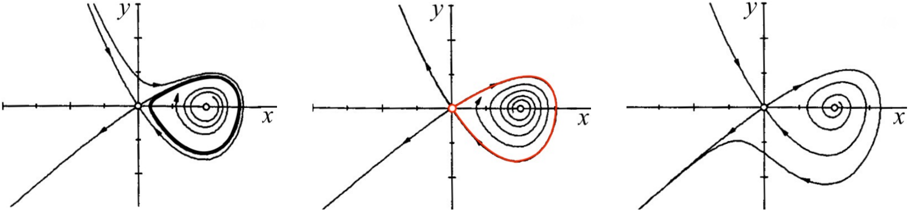
$\omega = 1$



$\omega < 1$

Case II C: Homoclinic bifurcation. Bif. parameter is μ and $\mu_c \approx -0.8645$.
Example system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu y + x - x^2 + xy \end{cases} \tag{9}$$



$\mu < \mu_c$ $\mu = \mu_c$ $\mu > \mu_c$

Conclusions

- Classification of bifurcations in 2-D systems
- Bifurcations of fixed points
- Bifurcations of closed orbits
- The supercritical vs. subcritical Hopf bifurcations
- Dangers associated with the *Hopf* bifurcation
- Hysteresis on the level of cycles

Revision questions

- Classification of bifurcations in 2-D.
- What is the Hopf bifurcation?
- What is the supercritical Hopf bifurcation?
- What is the subcritical Hopf bifurcation?
- What are global bifurcations of closed orbits?
- Name some global bifurcations of closed limit-cycles.
- What is a saddle-node coalescence (or bifurcation) of limit-cycles?
- What is hysteresis on the level of cycles?
- Name dangers associated with the *Hopf* bifurcation.
- What is a saddle-node infinite period bifurcation?
- What is a (saddle-loop or) homoclinic bifurcation?
- Name examples of dynamical instabilities.