Lecture №4: 2-D homogeneous linear systems, classification of fixed points in 2-D systems, the Lyapunov stability, basin of attraction

Dmitri Kartofelev, PhD

Tallinn University of Technology,
School of Science, Department of Cybernetics, Laboratory of Solid Mechanics

Lecture outline

- Phase portrait plotting.
- Eigenvalues and vectors of linear 2-D systems.
- Homogeneous 2-D systems and their solutions.
- Classification of fixed points in 2-D linear systems.
- Phase plane behaviour of aforementioned solutions about their fixed points.
- Saddles, nodes, spirals, centers, degenerate nodes, non-isolates fixed points.
- Basin of attraction.

Linear second-order differential equations

- Homogeneous differential equation:

$$a\ddot{x} + b\dot{x} + cx = 0 \quad \Leftrightarrow \quad \begin{cases} \dot{x} = y \\ a\dot{y} = -by - cx \end{cases} \tag{1}$$

- Nonhomogeneous differential equation:

$$a\ddot{x} + b\dot{x} + cx = f(x) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = y \\ a\dot{y} = -by - cx + f(x) \end{cases} \tag{2}$$

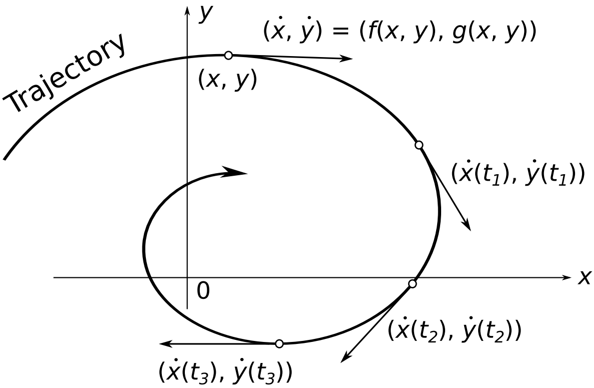
- Non-autonomous differential equation:

$$a\ddot{x} + b\dot{x} + cx = g(t) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = y \\ a\dot{y} = -by - cx + g(t) \end{cases} \tag{3}$$

In cases (1) and (2) a, b, and c are the constant coefficients.
In case (3) a, b, and c may depend on time t but don't have to.
Above functions f and g are arbitrarily selected and linear.

2-D phase portrait[1]

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{4}$$



[1]See Mathematica .nb file uploaded to course webpage.

Examples of 2-D systems

Harmonic oscillator[2] is given in dimensionless and normalised form

$$\ddot{x} + x = 0, \tag{5}$$

where x is the displacement.

Damped harmonic oscillator[2] is given in the following form:

$$\ddot{x} + x + \dot{x} = 0, \tag{6}$$

where x is the displacement and $\dot{x}$ term it the attenuation term.

[2]See Mathematica .nb file uploaded to course webpage.

2-D linear systems

Let's consider 2-D linear system given in the form

$$\dot{\vec{x}} = A\vec{x}, \tag{7}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{8}$$

$\{a, b, c, d\} \in \mathbb{R}$, and $\vec{x} = (x, y)^T$. The fixed point $\vec{x}^* = \vec{0}$ and the phase portrait near $\vec{x}^* = \vec{0}$ are fully determined by the eigenvalues and eigenvectors of system matrix A.

What are the eigenvalues and eigenvectors of a system? Following is a short reminder of your linear algebra courses.

2-D linear systems

We assume or seek a straight line solution in the following form:

$$\vec{x}(t) = \vec{v}\mathrm{e}^{\lambda t}, \tag{9}$$

where $\vec{v}$ is the eigenvector and λ is the eigenvalue.
Eqs. (7) and (9) yield

$$\dot{\vec{x}} = \underline{\vec{v}\lambda\mathrm{e}^{\lambda t}} \Leftrightarrow A\vec{x} = \underline{A(\vec{v}\mathrm{e}^{\lambda t})}, \tag{10}$$

$$\vec{v}\lambda\mathrm{e}^{\lambda t} = A\vec{v}\mathrm{e}^{\lambda t} \quad | \div \mathrm{e}^{\lambda t}, \tag{11}$$

$$\boxed{\lambda\vec{v} = A\vec{v},} \tag{12}$$

a useful relationship between the eigenvalues and eigenvectors of a system.

2-D linear systems

The (straight line) solution exists if one can find eigenvalues λ_i and eigenvectors $\vec{v}_i$.
Eigenvalue λ is given by

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \boxed{\lambda^2 - \tau\lambda + \Delta = 0}, \qquad (13)$$

where the boxed part is called the characteristic equation of a system and where

$$\tau = a + d \quad \text{is the trace of matrix } A \qquad (14)$$

and

$$\Delta = ad - bc \quad \text{is the determinant of matrix } A. \qquad (15)$$

2-D linear systems

$$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

Algebraic form of two eigenvalues λ_i is the following:

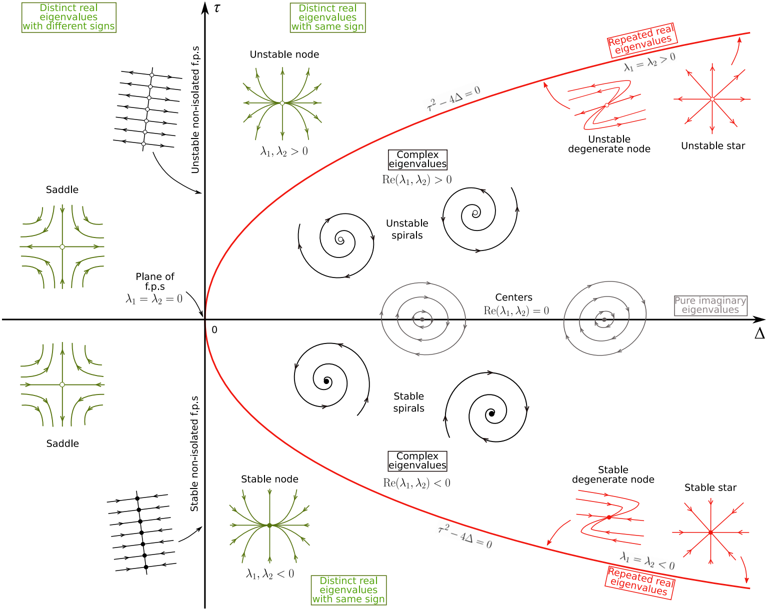
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}. \quad (16)$$

Additional *nice* properties of λ_i and Δ are the following:

$$\tau = \lambda_1 + \lambda_2, \quad (17)$$

$$\Delta = \lambda_1 \lambda_2. \quad (18)$$

Classification of fixed points in 2-D linear systems



Classification of fixed points: **Flowchart**

- if $\Delta < 0$:
  Isolated fixed point
  CASE 1: **Saddle point**
- if $\Delta = 0$:
  Non-isolated fixed points
  - if $\tau < 0$:
    CASE 5a: **Line of stable fixed points**
  - if $\tau = 0$:
    CASE 5b: **Plane of fixed points**
  - if $\tau > 0$:
    CASE 5a: **Line of unstable fixed points**

Continued on the next slide.

Classification of fixed points in 2-D linear systems

- if $\Delta > 0$:
  Isolated fixed point
  - if $\tau < -\sqrt{4\Delta}$:
    CASE 2a: **Stable node**
  - if $\tau = -\sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
      CASE 4a: **Stable degenerate node**
    - if there are no uniquely determined eigenvectors (both are non-unique):
      CASE 4b: **Stable star**
  - if $-\sqrt{4\Delta} < \tau < 0$:
    CASE 2b: **Stable spiral**
  - if $\tau = 0$:
    CASE 3: **Centre**
  - if $0 < \tau < \sqrt{4\Delta}$:
    CASE 2b: **Unstable spiral**
  - if $\tau = \sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
      CASE 4a: **Unstable degenerate node**
    - if there are no uniquely determined eigenvectors (both are non-unique):
      CASE 4b: **Unstable star**
  - if $\sqrt{4\Delta} < \tau$:
    CASE 2a: **Unstable node**

Basin of attraction

Basin of attraction of a fixed point (or an attractor) is the region of the phase space, over which integration (or iteration) is defined, such that any point (any initial condition) in that region will eventually be integrated into an attracting region (an attractor) or to a particular stable fixed point.

In the case of linear systems with a **stable fixed point**, every point in the phase space is in the basin of attraction of that system.

Conclusions

- Phase portrait plotting.
- Eigenvalues and vectors of linear 2-D systems.
- Homogeneous 2-D systems and their solutions.
- Classification of fixed points in 2-D linear systems.
- Phase plane behaviour of aforementioned solutions about their fixed points.
- Saddles, nodes, spirals, centers, degenerate nodes, non-isolates fixed points.
- Basin of attraction.

Revision questions

- How to plot a 2-D phase portrait of a system?
- What are 2-D homogeneous linear systems?
- What are non-homogeneous systems?
- Classification of fixed points in 2-D systems.
- Sketch a saddle node fixed point.
- Sketch a stable node fixed point.
- Sketch an unstable node fixed point.
- Sketch a stable spiral (fixed point).
- Sketch an unstable spiral (fixed point).
- Sketch a center (fixed point).
- Sketch a stable non-isolated fixed point.
- Sketch an unstable non-isolated fixed point.
- What are 2-D homogeneous nonlinear systems?
- What does it mean that a fixed point is Lyapunov stable?
- Give an example of Lyapunov stable fixed point.