Lecture №3: Over-damped bead on a rotating hoop, dimensionless form of equations of motion, introduction to 2-D systems, uniqueness of solution and phase space trajectories

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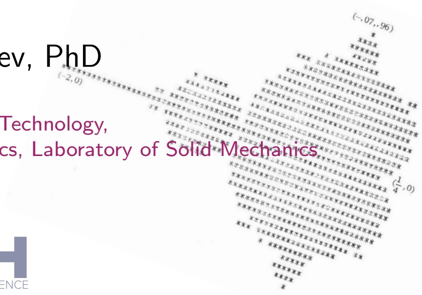

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Lecture outline

- Example from classical mechanics: the over-damped bead on a rotating hoop
- Dimensional analysis and dimensionless form of equations of motion
- 2-D systems/2nd order systems
- 2-D phase portrait
- Consequences of existence and uniqueness of solutions of 2-D linear systems on phase portrait trajectories

Ex: Over-damped bead on a rotating hoop

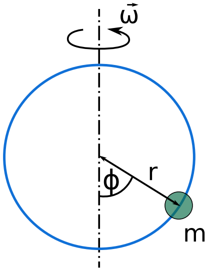


Figure: Scheme of the problem and used notations.

Derivation of the equation of motion in the following form:

$$mr\ddot{\phi} = -br\dot{\phi} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi). \tag{1}$$

Ex: Over-damped bead on a rotating hoop

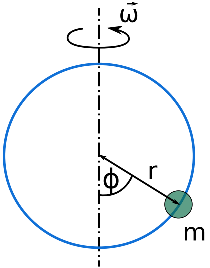
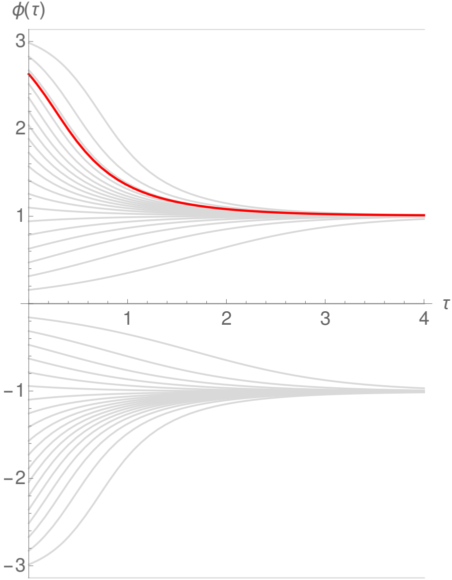


Figure: Scheme of the problem and used notations.

- Bifurcation analysis and construction of bifurcation diagram
- Dimensional analysis, non-dimensional normalised form

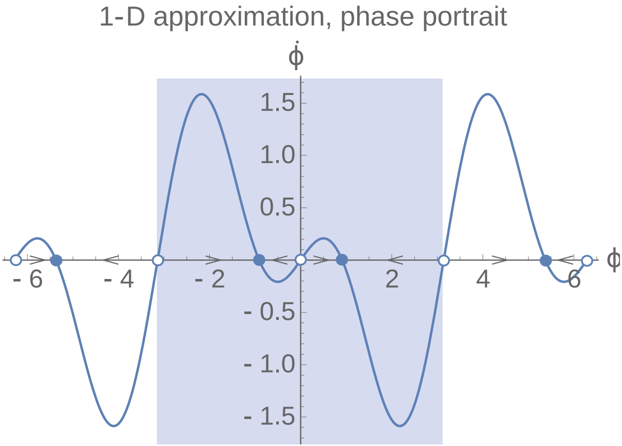
Ex: Over-damped bead on a rotating hoop



A family of solutions of 1-D problem in the form:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$$

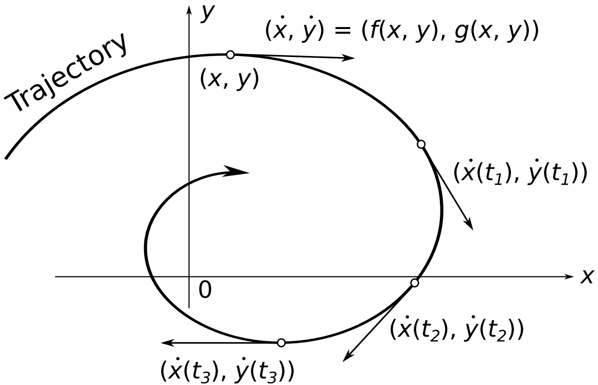
Ex: Over-damped bead on a rotating hoop



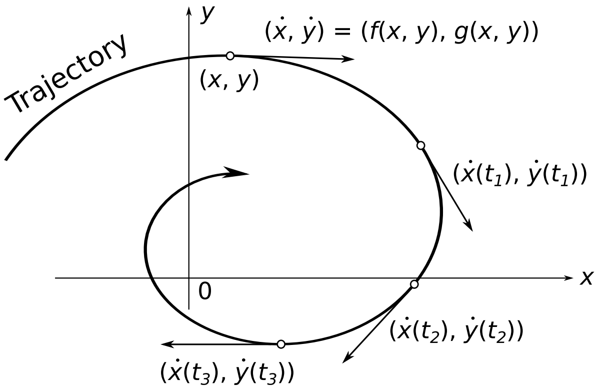
$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right) \qquad (2)$$

2-D phase portrait

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \qquad \text{or} \qquad \dot{\vec{x}} = \vec{f}(\vec{x}) \tag{3}$$



2-D systems: Existence and uniqueness



Existence and uniqueness: Solutions to $\dot{\vec{x}} = \vec{f}(\vec{x})$ *exist* and they are *unique* if $\vec{f}(\vec{x})$ and $\vec{f}'(\vec{x})$ are continuous. Function $\vec{f}$ is continuously differentiable.

2-D systems

What does **uniqueness** and existence of solutions imply?

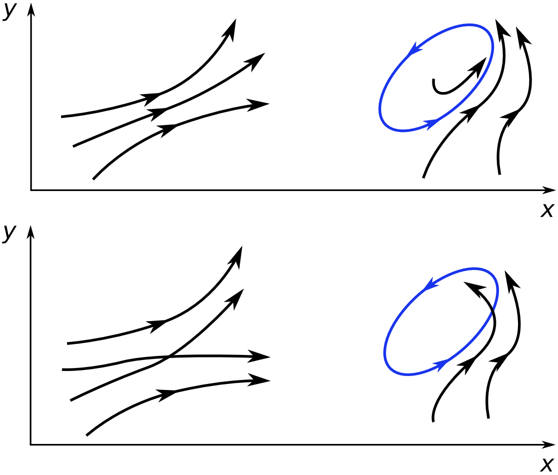


Figure: (Top) Possible dynamics. (Bottom) Impossible dynamics. The closed trajectory is showed with the blue closed curve.

2-D systems

What does **uniqueness** and existence of solutions imply?

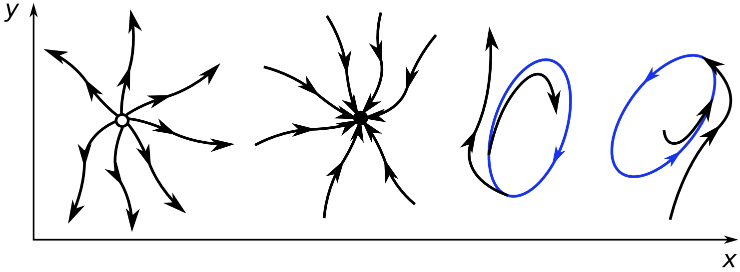
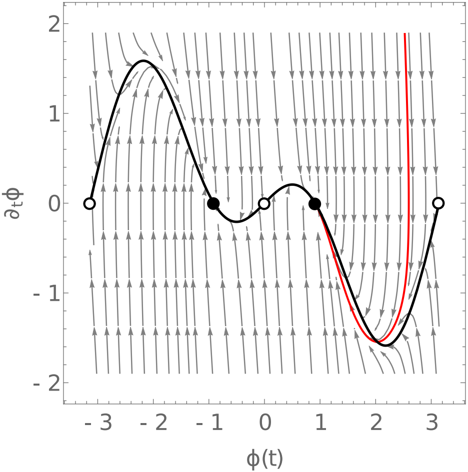


Figure: Additional possible dynamics: trajectories can approach or emanate from a fixed point (no motion at the fixed point), and they can merge with or emanate from closed trajectories (limit-cycles, only in nonlinear cases).

Ex: Over-damped bead on a rotating hoop



Phase portrait of 2-D problem:

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}\phi}{\mathrm{d}\tau}+\sin(\phi)(\gamma\cos\phi-1), \tag{4}$$

where

$$\alpha = \frac{m^2 g}{b^2 r}, \tag{5}$$

$$\gamma = \frac{r\omega^2}{g}. \tag{6}$$

1-D phase portrait of $\dfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\left(\gamma\cos\phi - 1\right)$ is shown with black.

Conclusions

- Example from classical mechanics: the over-damped bead on a rotating hoop
- Dimensional analysis and dimensionless form of equations of motion
- 2-D systems/2nd order systems
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Revision questions

- What is symmetry-broken solution?
- What is dimensional analysis of an equation of motion?
- What is dimensionless form of an equation?
- What is normalised form of an equation?
- How many initial conditions does first-order ODE have?
- How many initial conditions does second-order ODE have?
- Explain the notion of different time scales of a dynamical system.

Revision questions

- Derive the dimensionless form of the following equation of motion:
$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + ku = 0, \tag{7}$$
where u is the displacement, m is the mass and t is the time. Additionally, determine the dimensions of damping coefficient b and stiffness k.
- Derive the dimensionless form of the following nonhomogeneous equation of motion:
$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + ku = F_0\cos\omega_0 t, \tag{8}$$
where u is the displacement, m is the mass, F_0 and ω_0 are the driving force parameters, and t is the time. Additionally, determine the dimensions of damping coefficient b, stiffness k, driving force F_0 and driving force frequency ω_0.