Lecture №2: 1-D problems, linear analysis, bifurcation, bifurcation diagram

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Lecture outline

- Linearisation about a fixed point x^*
- Linear stability analysis of fixed point x^*
- Existence and uniqueness of solutions of 1-D systems
- Impossibility of oscillations in 1-D systems
- Bifurcation
- Bifurcation point (value of parameter)
- Saddle-node bifurcation. Ex: $\dot{x} = r \pm x^2$
  Ex: $\dot{x} = r + x - \ln(1 + x)$
- Bifurcation diagram
- Normal form
- Transcritical bifurcation. Ex: $\dot{x} = rx \pm x^2 = x(r \pm x)$.
- Pitchfork bifurcation. Ex: $\dot{x} = rx \pm x^3$.
- Supercritical ($\dot{x} = rx - x^3$) and subcritical ($\dot{x} = rx + x^3$) pitchfork bifurcations

Linearisation of 1-D systems

The one-dimensional system is given by

$$\dot{x} = f(x). \tag{1}$$

The dynamics close to fixed point x^* can be expressed as follows

$$x(t) = x^* + \eta(t), \tag{2}$$

where $|\eta| \ll 1$ is a small perturbation. The behaviour and change of solution x over time thus is

$$\dot{x} = (x^* + \eta)^{\cdot} = \dot{\eta}. \tag{3}$$

At the same time (1) holds. This mans that the dynamics of small perturbations is the following

$$\dot{\eta} = f(x) = f(x^* + \eta). \tag{4}$$

Linearisation of 1-D systems

Taylor series expansion about x^* of (4) results in

$$\begin{aligned}
\dot{\eta} &= f(x) = f(x^* + \eta) && (5)\\
&= f(x^*) + \frac{f'(x^*)}{1!}(x^* + \eta - x^*) + \frac{f''(x^*)}{2!}(x^* + \eta - x^*)^2 + \ldots && (6)\\
&= f'(x^*)\eta + \underbrace{\frac{f''(x^*)}{2!}\eta^2 + \ldots}_{\text{higher order terms, } O(\eta^2)} && (7)\\
&\approx f'(x^*)\eta. && (8)
\end{aligned}$$

If $f'(x^*) \neq 0$, then the term $|f'(x^*)\eta| \gg \left|\frac{f''(x^*)}{2!}\eta^2\right|$. Neglecting $O(\eta^2)$ yields the linearisation of the system about fixed point x^*

$$\dot{\eta} = s\eta, \tag{9}$$

where $s = f'(x^*)$ is simply the slope of function $f(x)$ evaluated at x^*.

Algebraic decay

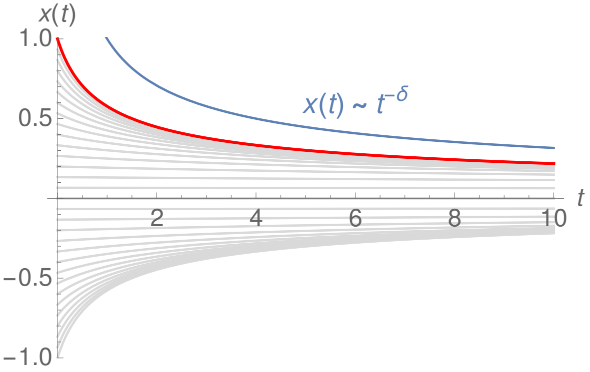
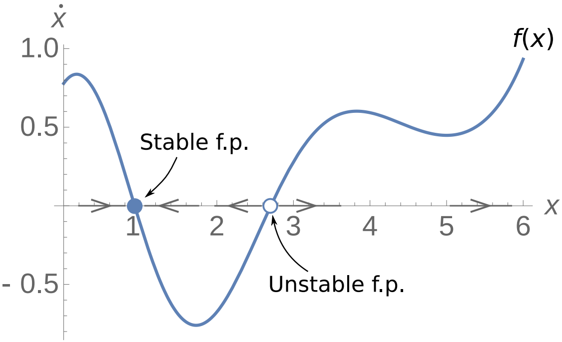


Figure: **Algebraic decay near a fixed point:** A family of numerical solutions of $\dot{x} = -x^3$. The initial condition of the solution shown with the red curve is $x(0) = 1$. An algebraic decay path $x(t) \sim t^{-\delta}$, where δ is constant, is shown with the blue curve.

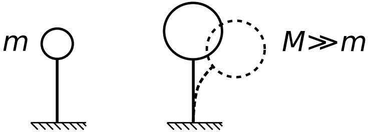
Existence and uniqueness



Existence and uniqueness: Solutions to $\dot{x} = f(x)$ *exist* and they are *unique* if $f(x)$ and $f'(x)$ are continuous, i.e, the function f is continuously differentiable.

Bifurcation

Bifurcation: The term is related to models with instabilities, sudden changes and transitions.



With the change of a parameter the qualitative structure of the *vector field* may change dramatically — fixed points may be created or destroyed, or they might change their stability. Such a change is called **bifurcation**.

Bifurcation point is the value of the parameter at witch the sudden change (bifurcation) occurs.

Bifurcation coordinate is the coordinate (free variable) at witch the bifurcation occurs.

Conclusions

- Linearisation about a fixed point x^*
- Linear stability analysis of fixed point x^*
- Bifurcation
- Bifurcation diagram
- Saddle-node bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation (subcritical, supercritical)

Revision questions

- What does linearisation of a nonlinear system imply?
- Linearise the following 1-D system

$$\dot{x} = x^3 - x \tag{10}$$

- What is bifurcation?
- What is bifurcation diagram?
- What is saddle-node bifurcation?
- What is transcritical bifurcation?
- What is pitchfork bifurcation?
- What is supercritical pitchfork bifurcation?
- What is subcritical pitchfork bifurcation?
- What is normal form in the context of bifurcations?
- Are oscillation possible in 1-D systems?
- Why are oscillations impossible in 1-D systems?
- What does uniqueness of solutions imply in the context of phase space trajectories?