Lecture №14: Fractals and fractal geometry, coastline paradox, spectral characteristics of dynamical systems, 1-D complex valued maps, the Mandelbrot set and nonlinear dynamical systems, introduction to applications of fractal geometry and chaos

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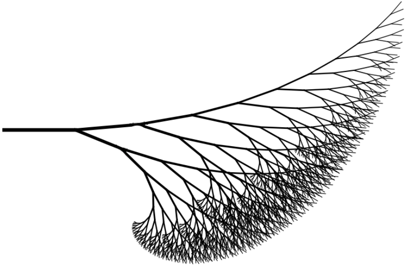
Lecture outline

- Definition of a fractal
- Spectral characteristics of periodic, quasi-periodic and chaotic systems
- Second look at 1-D and 2-D maps, complex valued maps
- The Mandelbrot set and the Fatou and Julia sets, their connection to nonlinear dynamical systems
- Generation of the Mandelbrot set and the corresponding Fatou sets
- The Buddhabrot
- The Multibrot sets
- Examples of fractal geometry in nature and applications
- Introduction to applications of fractals and chaos
    - Fractal similarity dimension and the coastline paradox
    - Synchronisation

Definition of a fractal[1]

Fractal **Endless** and complex *pattern* with fine structure at arbitrarily small scales. In other words magnification of **tiny features of a fractal are reminiscent of the whole**. Similarity can be exact (invariant), more often it is approximate or statistical.



Examples: The Cantor set, the von Kock curve, the Hilbert curve, the L-systems, etc.

[1]See Mathematica .nb file uploaded to the course webpage.

Spectral characteristics of dynamical systems

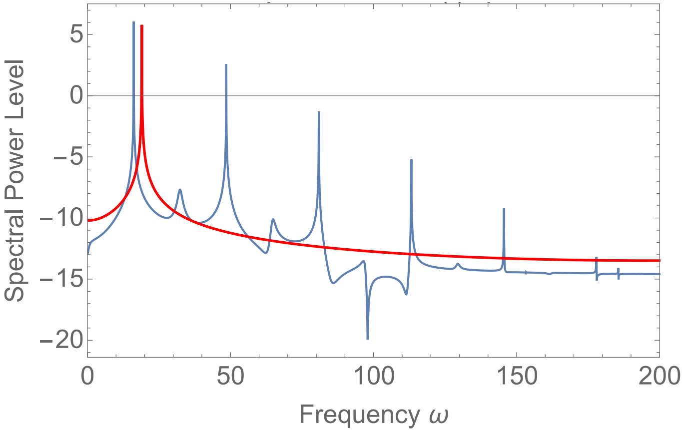


Figure: Sine wave shown with the red and a periodic solution of the Lorenz attractor shown with the blue.

Spectral characteristics of dynamical systems

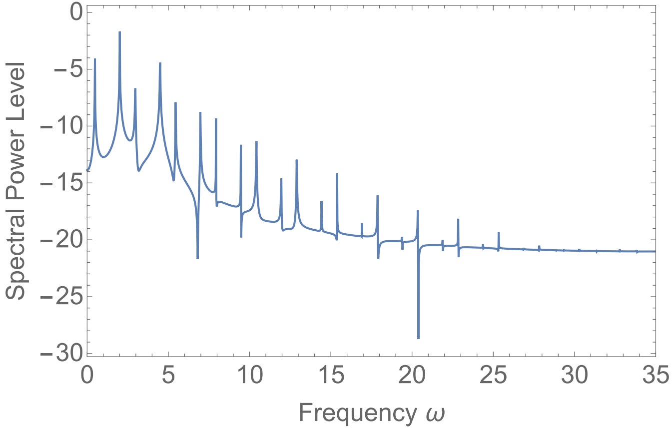


Figure: Quasi-periodic solution.

Spectral characteristics of dynamical systems

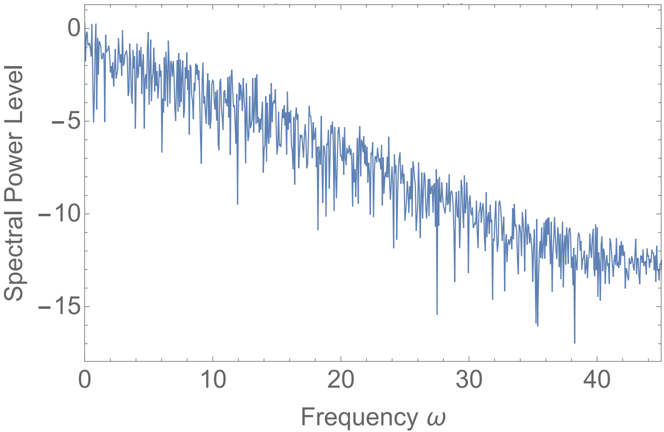
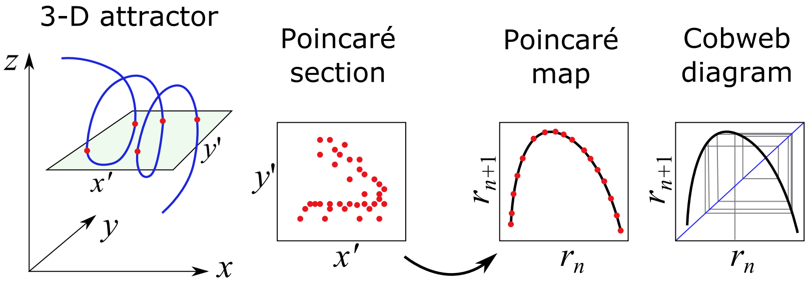


Figure: Chaotic solution.

Dynamics analysis methods

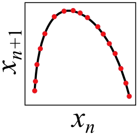


$$\begin{cases} x'_{n+1} = f_1(x'_n, y'_n) \\ y'_{n+1} = f_2(x'_n, y'_n) \end{cases} \Rightarrow \quad r_{n+1} = f_3(r_n) \qquad (1)$$

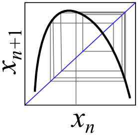
Construction of the Poincaré map $\vec{P}(\vec{x}') = (f_1(x', y'), f_2(x', y'))^T$ (1). Mapping of the Poincaré section points where r is the radial distance from the origin (in the case of a "flat" attractor).

Dynamics analysis methods

Poincaré map



Cobweb diagram



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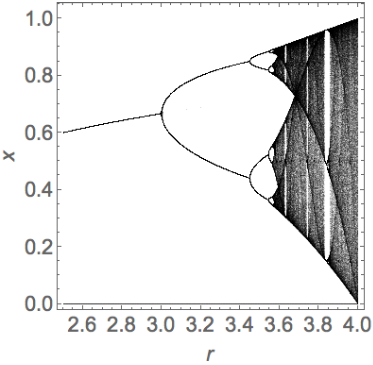
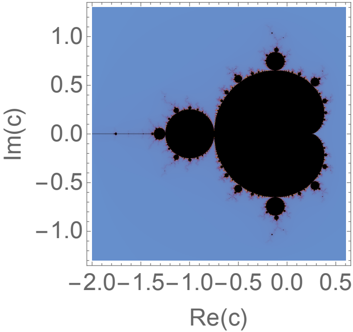


Orbit diagram — a long-term discrete-time behaviour analysis.

The Mandelbrot set and dynamical systems

The Mandelbrot set[2] M is defined as follows

$$\begin{cases} z_{n+1} = z_n^2 + c, \quad \{z, c\} \in \mathbb{C},\ n \in \mathbb{Z}^+ \\ z_0 = 0 \\ c \in M \iff \limsup\limits_{n\to\infty} |z_n| \le 2 \end{cases} \tag{2}$$



[2]See Mathematica .nb file uploaded to the course webpage.

The Mandelbrot set

The complex square map given in the form

$$z_{n+1} = z_n^2 + c, \tag{3}$$

where $z = x + \mathrm{i}y$, $c = r + \mathrm{i}s$, and z, $c \in \mathbb{C}$, can be represented as a 2-D real valued map. The component form of (3) is the following:

$$x_{n+1} + \mathrm{i}y_{n+1} = (x_n + \mathrm{i}y_n)^2 + r + \mathrm{i}s, \tag{4}$$

$$x_{n+1} + \mathrm{i}y_{n+1} = x_n^2 + 2\mathrm{i}x_n y_n - y_n^2 + r + \mathrm{i}s, \tag{5}$$

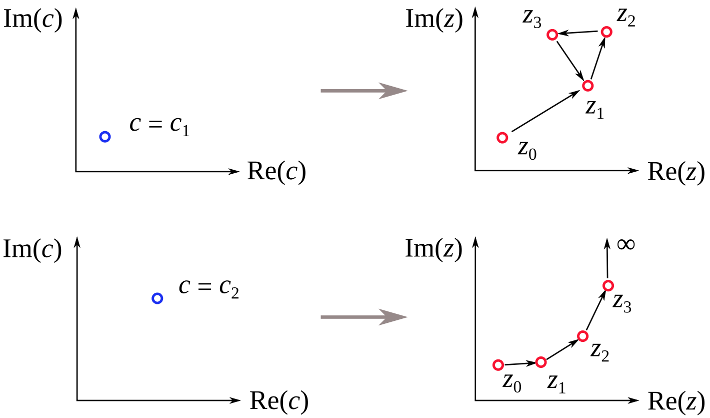
$$x_{n+1} + \mathrm{i}y_{n+1} = x_n^2 - y_n^2 + r + \mathrm{i}(2x_n y_n + s). \tag{6}$$

Separation of the real and imaginary parts, and elimination of the imaginary unit i yields

$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 + r \\ \not{\mathrm{i}}y_{n+1} = \not{\mathrm{i}}(2x_n y_n + s) \end{cases} \Rightarrow \begin{cases} x_{n+1} = x_n^2 - y_n^2 + r, \\ y_{n+1} = 2x_n y_n + s, \end{cases} \tag{7}$$

where x, y, r, $s \in \mathbb{R}$.

1-D complex maps, non-trivial dynamics



Fixing the polynomial

System dynamics

The Mandelbrot set, self-similar properties (video)

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The Fatou sets and dynamical systems

The Fatou set F_c corresponding to the M set with fixed c value is defined as follows

$$\begin{cases} z_{n+1} = z_n^2 + c, \quad \{z, c\} \in \mathbb{C},\ n \in \mathbb{Z}^+ \\ c = \text{const.} = |c| \leq 2 \\ z_0 \in F_c \iff \limsup_{n\to\infty} |z_n| \leq 0.5 + \sqrt{0.25 - |c|} \end{cases} \tag{8}$$

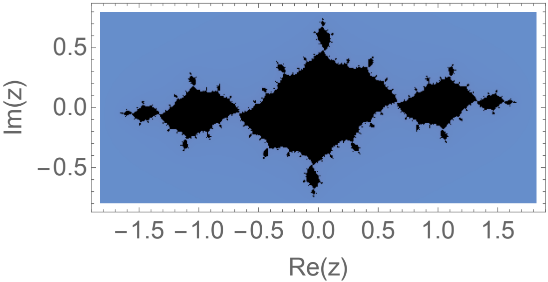


Figure: The Fatou set or the filled Julia set where $c = -1.1 - 0.1\mathrm{i}$.

The Julia sets

The Julia set J_c corresponding to the M set with fixed c value is defined as follows.

Definition: The Julia set contains the compact boundary of a nonempty Fatou set.

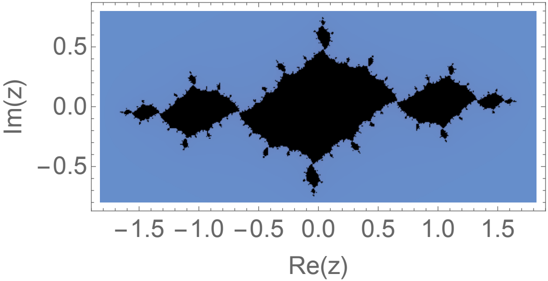
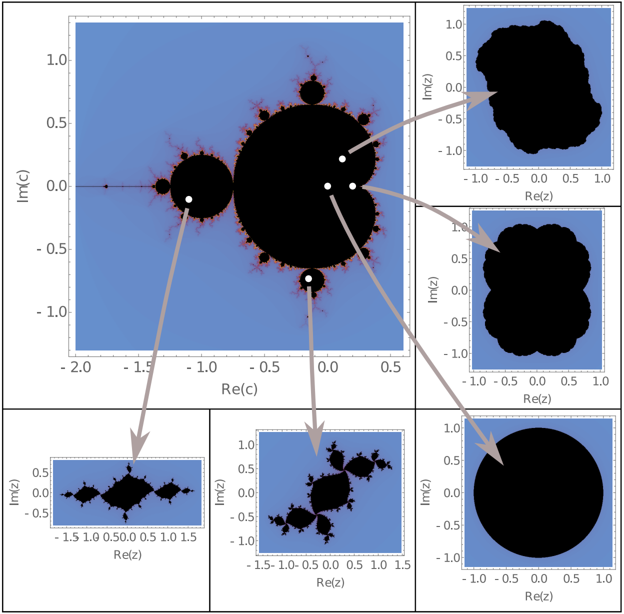
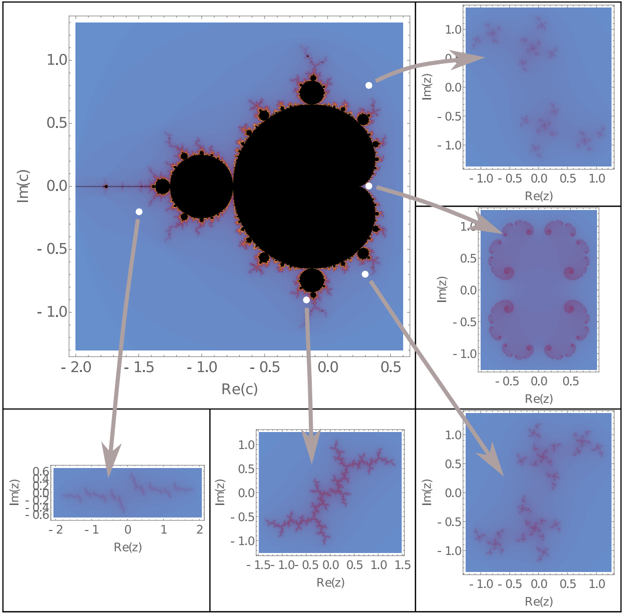


Figure: The Julia set where $c = -1.1 - 0.1\mathrm{i}$. The set is the boundary between the black and blue colours.

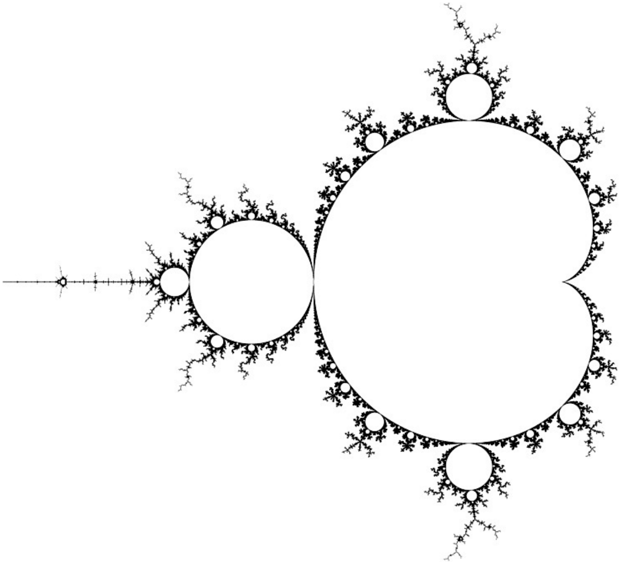
The Mandelbrot set and the Fatou sets



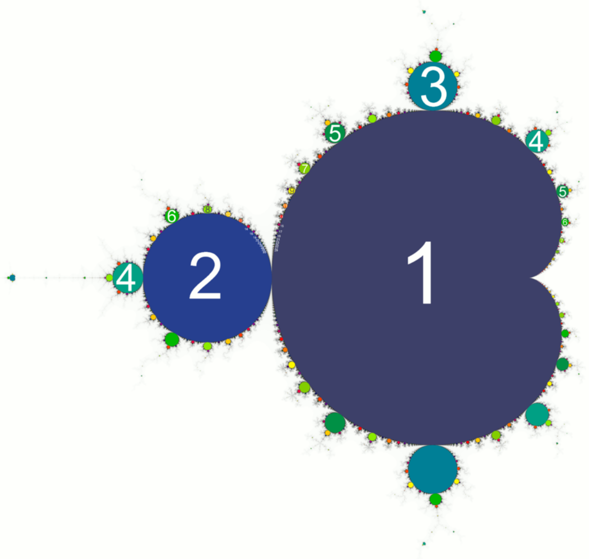
The Mandelbrot set and the Fatou sets/Fatou dust



Fractal dimension of the edge $d = 2.0$



The Mandelbrot set and period-p orbits



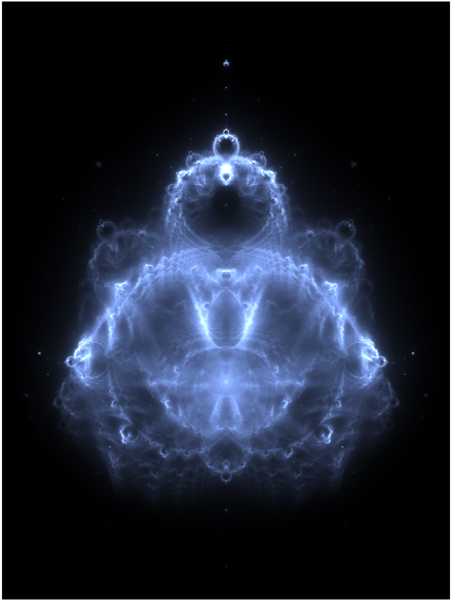
Credit: CC BY-SA 3.0 Hoehue commonswiki

The main cardioid

Certain caustics can take the shape of a cardioid.

Figure: Caustic in a coffee cup.

The Mandelbrot set, the Buddhabrot

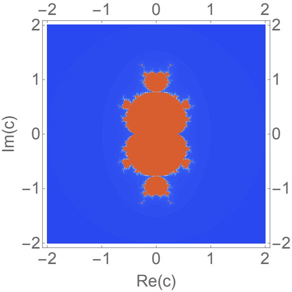


The image is rotated in a clockwise direction by 90°.

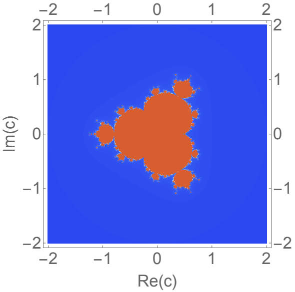
Credit: CC BY-SA 3.0 Purpy Pupple, Evercat, Michael Pohoreski

Generalised Mandelbrot sets, Multibrot sets[3]

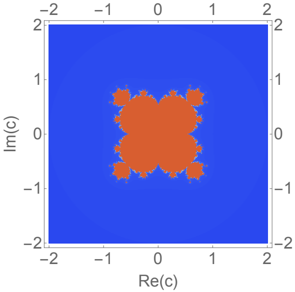
$$z_{n+1} = z_n^p + c, \quad z_0 = 0 \tag{9}$$



$p = 3,$



$p = 4$



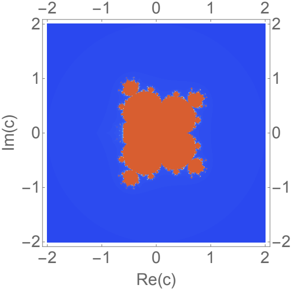
$p = 5$

The main bulb is nephroid.

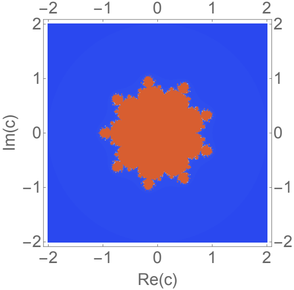
[3] See Mathematica .nb file uploaded to the course webpage.

Generalised Mandelbrot sets, Multibrot sets

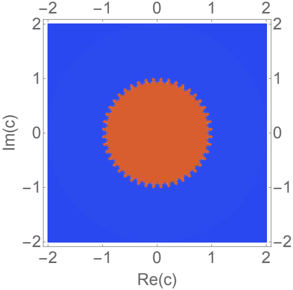
$$z_{n+1} = z_n^p + c, \quad z_0 = 0$$



$p = 5.5$



$p = 10$



$p = 45$

Fractal geometry and nature

Fractal geometry and nature

Yarlung Tsangpo River, China. Credit: NASA/GSFC/LaRC/JPL, MISR Team.

Fractal geometry and nature

Fractal geometry and nature

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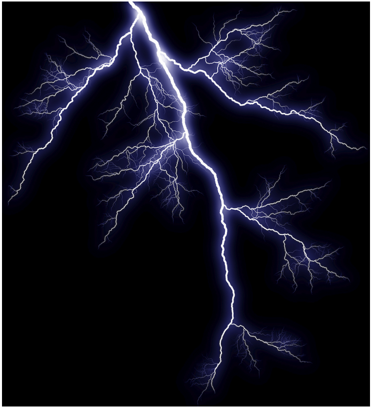
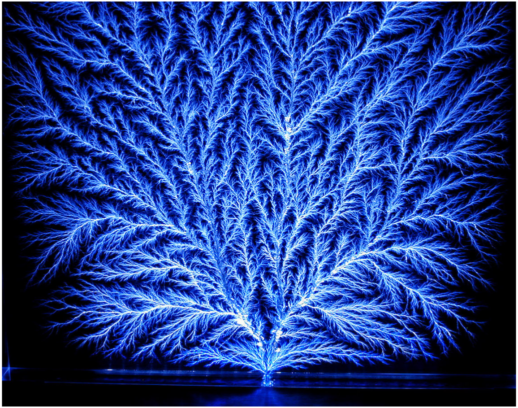
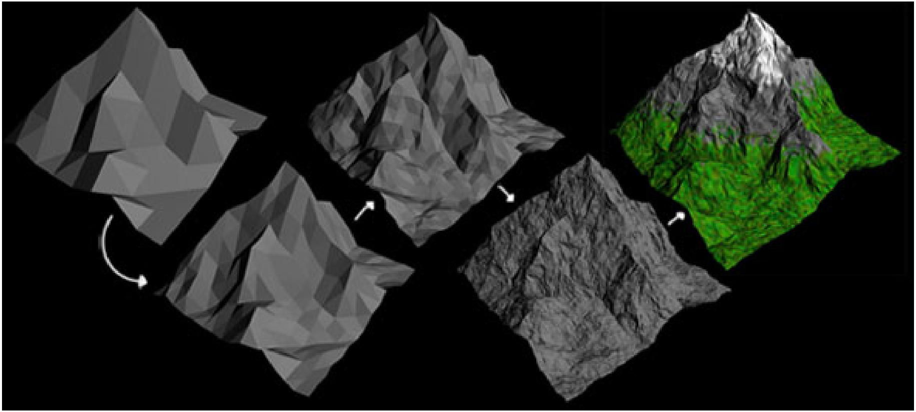
Fractal geometry and nature

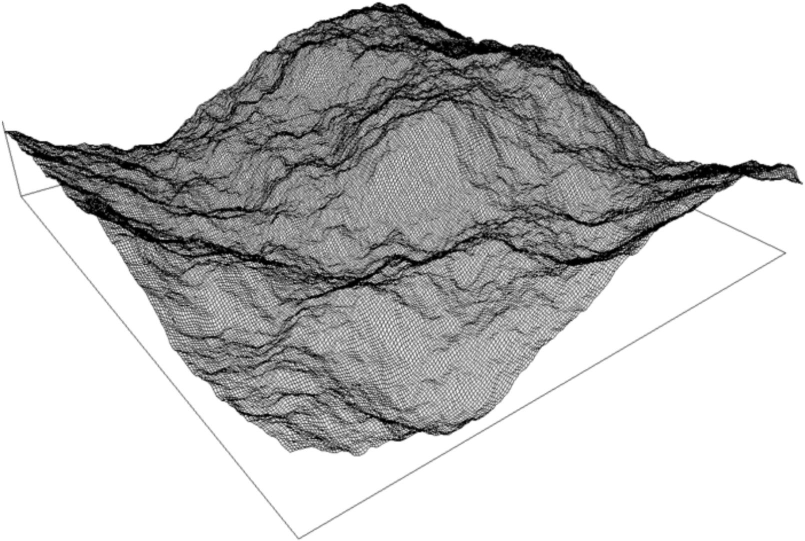




Fractal geometry and nature (computer graphics)

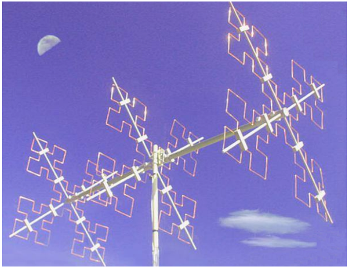


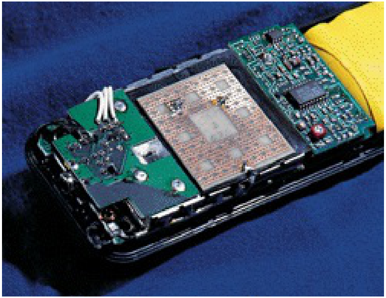
Fractal geometry and nature (computer graphics)

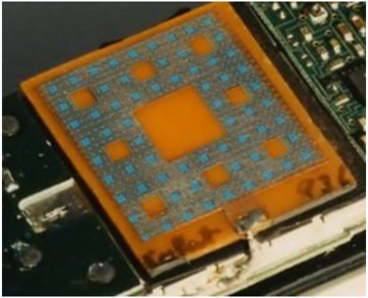


Brownian noise with $d = 2.0 \rightarrow$ topography.

Fractal geometry and technology







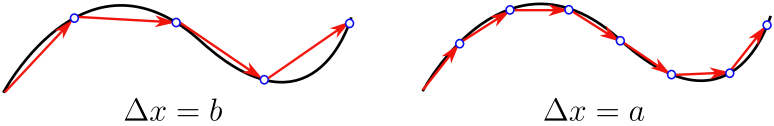
The coastline paradox

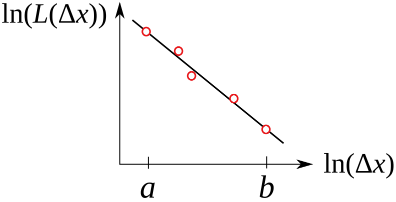
The **coastline paradox**[4] is the counterintuitive observation that the coastline of a landmass does not have a well-defined length. This results from the **fractal-like properties of coastlines**. The first recorded observation of this phenomenon was by Lewis Fry Richardson and it was expanded by Benoit Mandelbrot.

Read: B. Mandelbrot, "How long is the coast of Britain? Statistical self-similarity and fractional dimension," *Science, New Series*, **156**(3775), 1967, pp. 636–638.

[4] See Mathematica .nb file uploaded to the course webpage.

The coastline paradox





Slope of the resulting graph

$$d = -\frac{\ln(L(\Delta x))}{\ln(\Delta x)} = \frac{\ln(L(\Delta x))}{\ln(1/\Delta x)}, \quad (10)$$

where L is the resulting measurement and Δx is the measurement resolution, i.e., length of the measuring stick.

The coastline paradox

Google

length of coastline of uk

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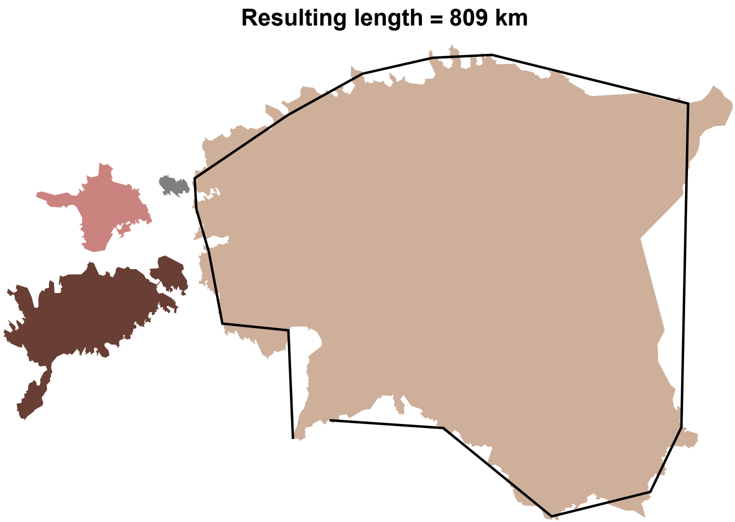
Coastline of the United Kingdom / Length

around 12,429 km

If the larger islands are added the coastline, as measured by the standard method at Mean High Water Mark, rises to **about 19,491 miles (31,368 km)**. According to the CIA Factbook, the length of the UK coastline is **around 12,429 km** or **7,723 miles**, although no details are provided about how this figure was calculated.

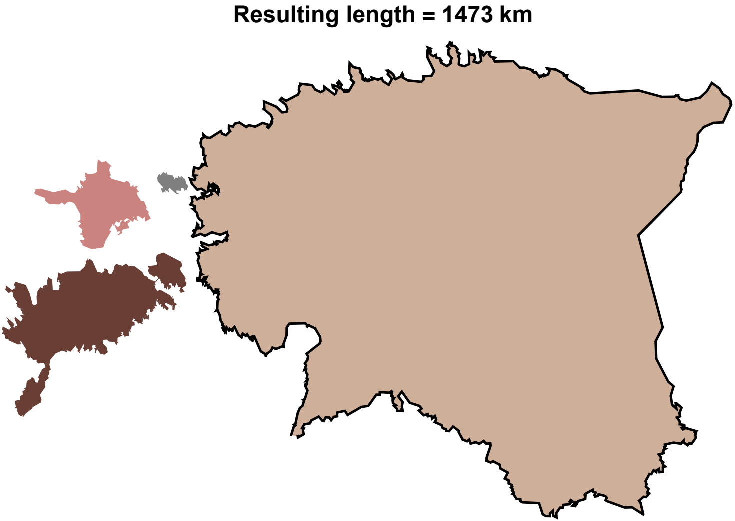
Coastline of the United Kingdom - Wikipedia
https://en.wikipedia.org/wiki/**Coastline**_of_the_**United_Kingdom**

The coastline paradox, Estonia[5]

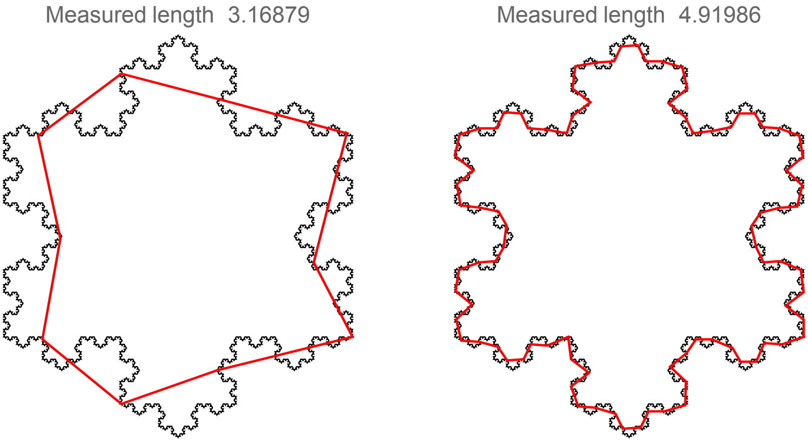


[5]See Mathematica .nb file uploaded to the course webpage.

The coastline paradox, Estonia

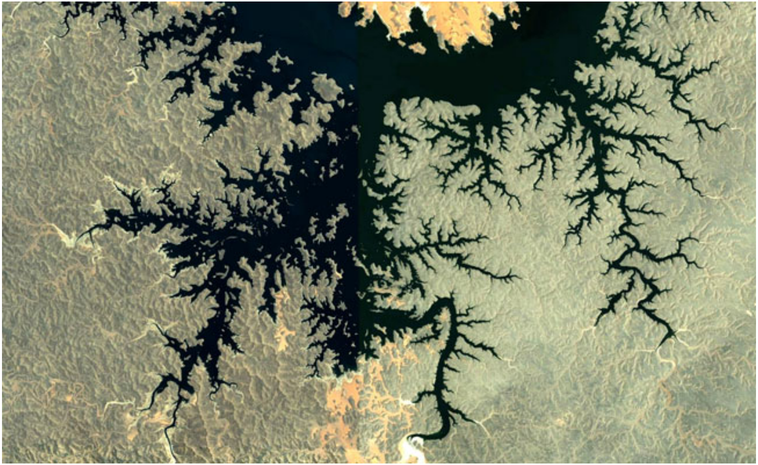


The coastline paradox, the von Kock snowflake[6]



[6]See Mathematica .nb file uploaded to the course webpage.

The coastline paradox



Great Britain $d = 1.25$; Norway $d = 1.52$; Estonia* $d = 1.2$; South African coast $d = 1.0$

Synchronisation: metronomes

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Synchronisation: fireflies

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Synchronisation: The Millennium bridge (2000)

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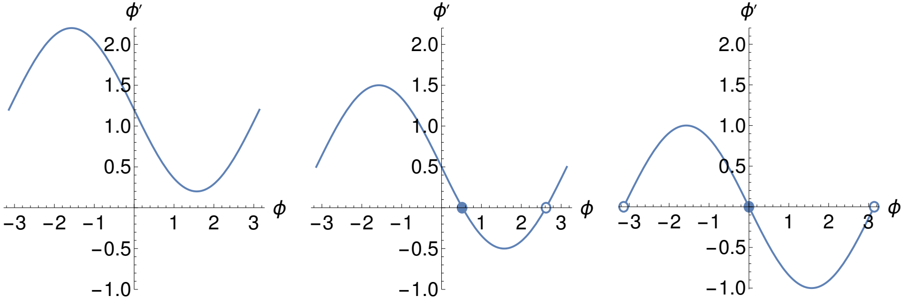
Synchronisation

Aperiodicity of chaos $\rightarrow$ bifurcation/s $\rightarrow$ periodic solution

Conceptual model

$$\dot{\phi} = \mu - \sin \phi, \tag{11}$$

where $\mu \geq 0$ is the system parameter and ϕ is the phase difference/s.



$\mu > 1$ $\quad$ $0 < \mu < 1$ $\quad$ $\mu = 0$

Conclusions

- Definition of a fractal
- Spectral characteristics of periodic, quasi-periodic and chaotic systems
- Second look at 1-D and 2-D maps, complex valued maps
- The Mandelbrot set and the Fatou and Julia sets, their connection to nonlinear dynamical systems
- Generation of the Mandelbrot set and the corresponding Fatou sets
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Revision questions

- Define fractal (technical definition).
- Define pre-fractal.
- Explain the coastline paradox.
- Can a coastline be described with Euclidean geometry?
- What determines spectral characteristics of dynamical systems?
- What is a 1-D complex valued map?
- What are the Mandelbrot set and the Fatou sets?
- What is the Julia set?
- Assuming $z = x + \mathrm{i}y$, $c = r + \mathrm{i}s$, and $z, c \in \mathbb{C}$, show that map in the form
$$\begin{cases} x_{n+1} = x_n^2 - y_n^2 + r, \\ y_{n+1} = 2x_n y_n + s, \end{cases} \tag{12}$$
is the real counterpart of the Mandelbrot set.

Revision questions

- What is the physical meaning of the Mandelbrot set?
- What is the physical meaning of the Fatou sets?
- What is the generalised Mandelbrot set also known as the Multibrot set?
- Name an example of self-similar phenomena in nature.