Lecture №13: Basin of attraction, sensitive dependence on initial conditions, linearisation of 2-D maps, classification of fixed points in 2-D maps, linear analysis of the Hénon map

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TAL TECH
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Lecture outline

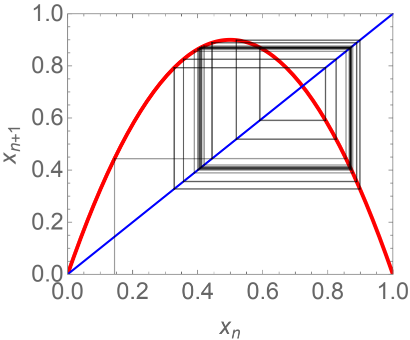
- Stretching–folding–re-injection dynamics in cobweb diagrams
- Sensitive dependence on initial conditions
- Basin of attraction
- Linearisation about fixed point $\vec{x}^*$
- Stability of fixed points and period-p points in 2-D maps
- Classification of fixed points in 2-D maps
- Linear analysis of fixed points of the Hénon map
- Geometry and dynamics of nonlinear maps near fixed points and period-p points
- Video feedback effect

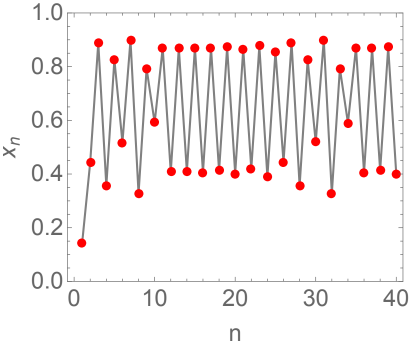
Mixing-folding dynamics in cobweb diagrams

We consider the logistic map

$$x_{n+1} = rx_n(1 - x_n), \quad (1)$$

where r is the control parameter.

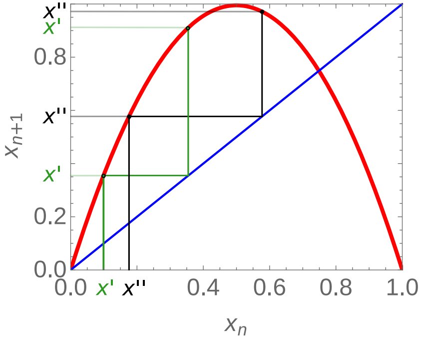




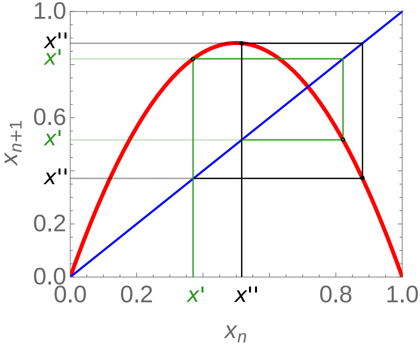
Can we see the stretching–folding–re-injection dynamics in chaotic cobweb plots?

Mixing-folding dynamics in cobweb diagrams

Dynamics of the interval $x_0 = [x', x'']$.



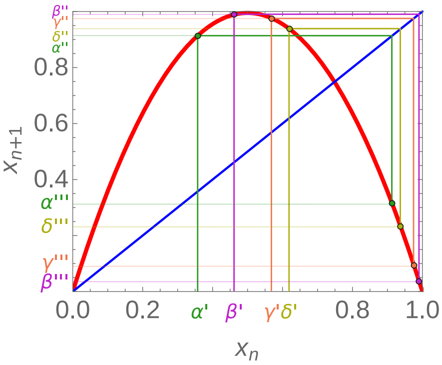
Stretching and squeezing of x_0.



Mixing and folding of x_0.

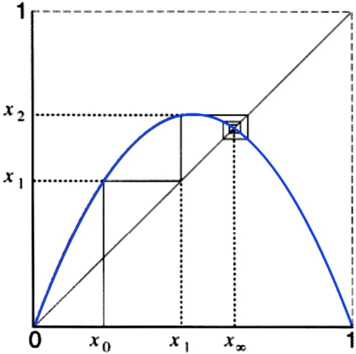
Mixing-folding dynamics in cobweb diagrams

Dynamics of selected points $\{\alpha, \beta, \gamma, \delta\} \in x_0$.



$$\{\alpha', \beta', \gamma', \delta'\} \underset{\text{mixing}}{\rightarrow} \{\alpha'', \delta'', \gamma'', \beta''\} \underset{\text{folding}}{\rightarrow} \{\beta''', \gamma''', \delta''', \alpha'''\}$$

Sensitive dependence on initial conditions[1]



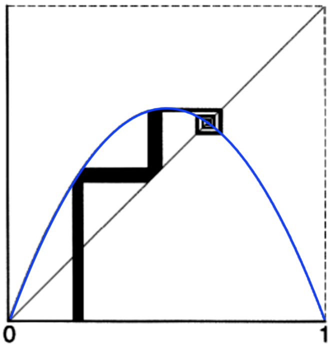


Figure: The logistic map where $1 < r < 3$. (Left) Iteration of a single point. (Right) Iteration of an interval. **No sensitivity** on initial condition for stable fixed point or period-p points.

[1] H. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, New York: Springer-Verlag, 2004 pp. 471–473

Sensitive dependence on initial conditions[2]

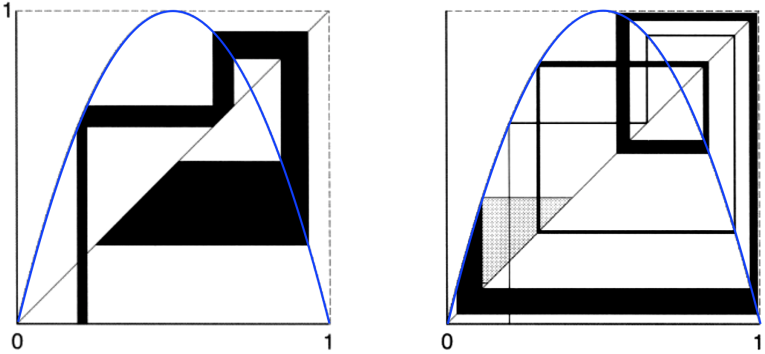


Figure: The logistic map where $r = 4$. (Left) Iteration of an interval of initial conditions. (Right) Iteration of an even smaller interval, also leading to large deviations.

[2] H. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, New York: Springer-Verlag, 2004 pp. 471–473

Sensitive dependence on initial conditions[3]

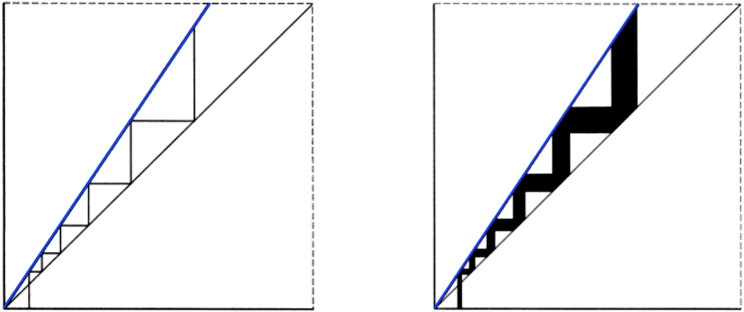


Figure: Map given by $x_{n+1} = cx_n$ where the constant $c > 1$. (Left) A single initial condition. (Right) An interval of initial conditions. The property of sensitivity to initial conditions **is central to chaos**. **Sensitivity, however, does not automatically lead to chaos in maps.**

[3] H. Peitgen, H. Jürgens, D. Saupe, *Chaos and Fractals: New Frontiers of Science*, New York: Springer-Verlag, 2004 pp. 471–473

Basin of attraction and a 2-D map

Example[4]: Determine the basin of attraction of the following nonlinear 2-D map given in polar coordinates

$$\begin{aligned} r_{n+1} &= r_n^2, \\ \theta_{n+1} &= \theta_n - \sin\theta_n. \end{aligned} \tag{2}$$

[4]See Mathematica .nb file uploaded to the course webpage.

2-D linear maps

Remainder from Lecture 4:

The (straight line) solution exists if one can find the λ's and $\vec{v}$'s. λ is given by

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \boxed{\lambda^2 - \tau\lambda + \Delta = 0}, \tag{3}$$

where the boxed part is called the characteristic equation of the system and where

$$\tau = a + d \quad \text{is the trace of matrix } A \tag{4}$$

and

$$\Delta = ad - bc \quad \text{is the determinant of matrix } A. \tag{5}$$

2-D linear maps

$$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

Algebraic form of λ is

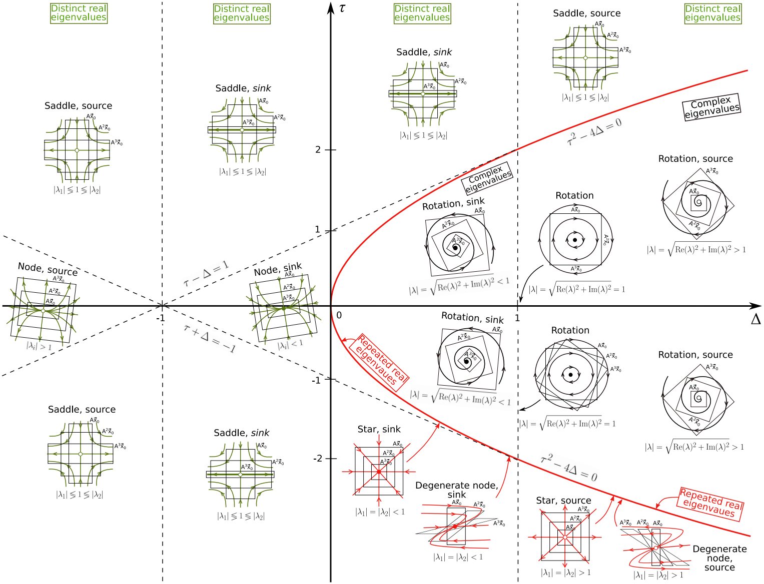
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}. \tag{6}$$

Additional *nice* propertied of λ and Δ are the following:

$$\tau = \lambda_1 + \lambda_2, \tag{7}$$

$$\Delta = \lambda_1 \lambda_2. \tag{8}$$

Classification of fixed points in 2-D linear maps



Classification of fixed points in 2-D linear maps

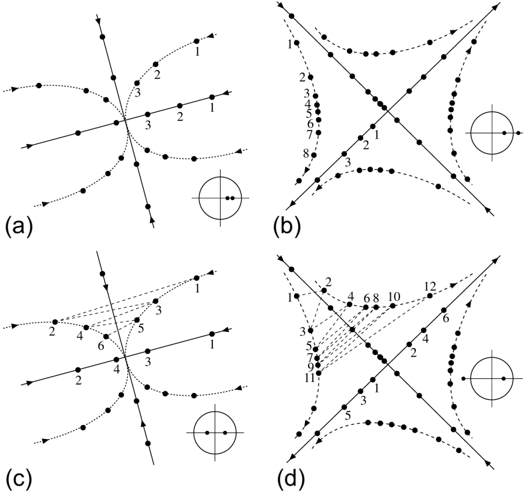


Figure: Phase portraits where the eigenvalues are distinct and real: (a) and (c) stable nodes, (b) and (d) saddles.

Negative eigenvalue forces the iterates to jump back and forth across the manifold (defined by the other eigendirection).

Image: A. Medio, M. Lines, *Nonlinear Dynamics: A Primer*, Cambridge University Press, 2001, p. 49

Linear analysis of the Hénon map

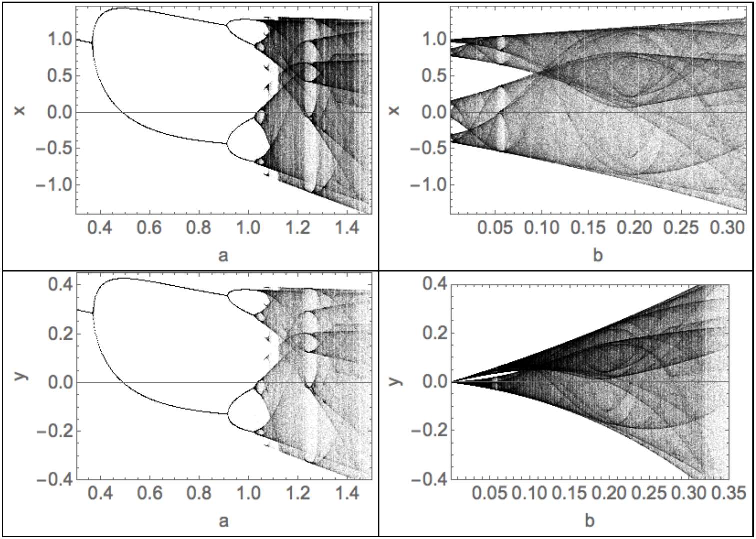
The Hénon map[5] is defined by

$$\begin{cases} x_{n+1} = 1 + y_n - a x_n^2, \\ y_{n+1} = b x_n, \end{cases} \tag{9}$$

where a and b are the parameters.

[5]See Mathematica .nb file uploaded to the course webpage.

The Hénon map dynamics[6]



[6] See Mathematica .nb file uploaded to the course webpage.

A chaotic map

Example[7]: A chaotic attractor in a 2-D map. The attractor is given in polar coordinates and in the following form:

$$
\begin{aligned}
r_{n+1} &= \sqrt{r_n}, \\
\theta_{n+1} &= 2\theta_n.
\end{aligned}
\tag{10}
$$

[7] See Mathematica .nb file uploaded to the course webpage.

Video feedback effect

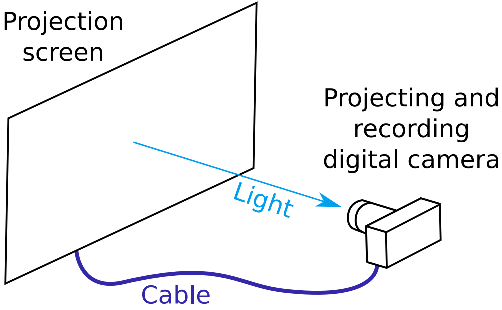


Figure: Basic setup for demonstrating the video feedback effect.

Video feedback effect

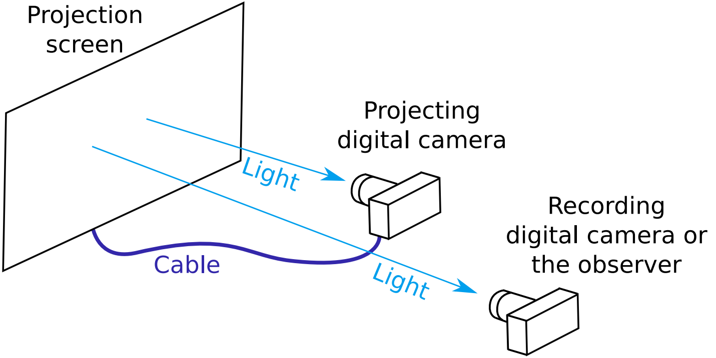


Figure: The setup in the following video.

Video feedback effect: 2-D mapping

Simplified (wrong) model of the effect.

$$\vec{F} \doteq \{c(x,y) \to c(f(x,y), g(x,y))\}, \tag{11}$$

or equivalently

$$c_{n+1}(x,y) = c_n(f(x,y), g(x,y)), \tag{12}$$

where continuous f and g are discretised:

$$\{x[i], y[j]\}, \text{ where } i,j \in \mathbb{Z},\ i \in [1, W],\ j \in [1, H]. \tag{13}$$

The projected image is composed of uniformly placed pixels $c[i,j]$. The total number of pixels is $W \cdot H$. n is the image frame number (iterate of the frame). Each pixel is assigned a colour depth value c (e.g. some 24 bit colour map, 8 bits per RGB channel totalling $2^{24} = 16\,777\,216$ colours). The functions f and g encompass all system dynamics and settings (camera, optics, electronics, digital signal processing, digital image transformations, etc.).

Video feedback effect: 2-D mapping

Alternative notation (more familiar to us). Component form:

$$\begin{cases} x^c_{n+1} = f(x^c_n, y^c_n) \\ y^c_{n+1} = g(x^c_n, y^c_n) \end{cases}, \quad \text{here} \quad x^c_n \in [1, W] \text{ and } y^c_n \in [1, H], \tag{14}$$

where each pixel (x, y) is assigned a colour c and each coordinate pair is sharing a colour, and n is the iterate (frame).

Matrix form:

$$\vec{c}_{n+1} = A\vec{c}_n, \qquad \vec{c} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{15}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and } \{a, b, c, d\} \in \mathbb{R}, \tag{16}$$

or in the nonlinear case

$$A = \begin{pmatrix} f_{1,1}(x_n, y_n) & f_{1,2}(x_n, y_n) \\ f_{2,1}(x_n, y_n) & f_{2,2}(x_n, y_n) \end{pmatrix}, \qquad f_{i,j} \text{ are the functions.} \tag{17}$$

Video feedback effect: video

No embedded video files in this pdf

Video feedback effect

What is/are the source/sources of the nonlinearity?
Hint: Compare to the optical feedback between mirrors.

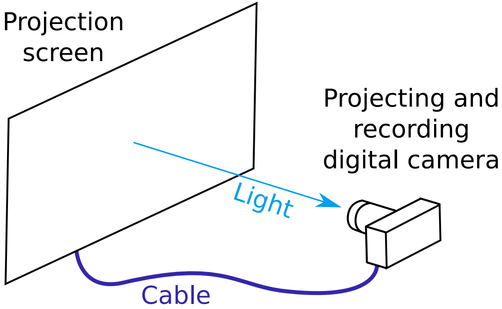


Figure: Basic setup for demonstrating the video feedback effect.

Conclusions

- Stretching–folding–re-injection dynamics in cobweb diagrams
- Sensitive dependence on initial conditions
- Basin of attraction
- Linearisation about fixed point $\vec{x}^*$
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Revision questions

- Is it possible to see stretching–folding–re-injection dynamics in cobweb plots?
- What does linearisation of a nonlinear 2-D map imply?
- Define sensitive dependence on initial conditions in maps.
- Define basin of attraction of a map.
- Sketch a saddle fixed point.
- Sketch a stable node (sink) fixed point.
- Sketch an unstable node (source) fixed point.
- What are improper oscillations of map iterates?
- What is the cause of improper oscillation of map iterates in terms of eigenvalues?
- What is the video feedback effect?