Lecture 8: Quasi-periodicity, 3-D and higher order systems, 3-D limit-cycle, introduction to chaos (deterministic chaos, chaos theory), chaotic water wheel, the Lorenz attractor, *coursework requirements*

Contents

Handout: Coursework requirements.
Demonstration: Trefoil knot, cinquefoil knot.

1 Quasi-periodicity

1.1 A graphical way of thinking about coupled oscillations

Besides the plane, another important two-dimensional phase space is the **torus**. It is the natural phase space for systems of the form

$$\begin{cases} \dot{\theta}_1 = f_1(\theta_1, \theta_2), \\ \dot{\theta}_2 = f_2(\theta_1, \theta_2), \end{cases} \tag{1}$$

where θ_1 and θ_2 are the angular displacements, and functions f_1 and f_2 are periodic in both arguments. An intuitive way to think about Sys. (1) is to imagine two friends jogging on a circular stadium—our approach so far. Here, $\theta_1(t)$ and $\theta_2(t)$ represent their angular positions with respect to the centre of the stadium. This polar coordinate representation is shown in Fig. 1.

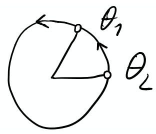


Figure 1: Circular stadium where the angular positions of two runners are shown with $\theta_1(t)$ and $\theta_2(t)$.

Another way to think about Sys. (1) is to imagine a point having coordinates $\theta_1(t)$ and $\theta_2(t)$ on a surface of a **torus** as shown in Fig. 2. But since the curved surface of a torus makes it hard to draw phase portraits, we prefer to use an equivalent representation: a square with **2π-periodic boundary conditions**. Then if a trajectory runs off an edge, it magically reappears on the opposite edge, see Fig. 2.

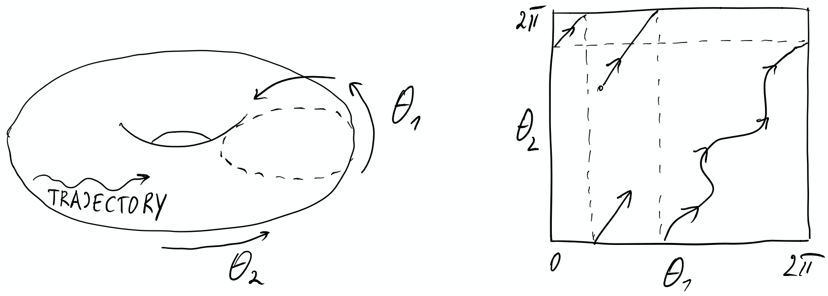


Figure 2: (Left) A trajectory of Sys. (1) on the surface of a torus. Coordinate system (θ_1, θ_2) for the torus. (Right) Equivalent representation of the torus surface where the plane is 2π-periodic both in θ_1 and θ_2.

1.2 Example: Decoupled case $K_1 = K_2 = 0$

We consider an example in the form

$$\begin{cases} \dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1), \\ \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2), \end{cases} \tag{2}$$

where θ_1 and θ_2 are the angular displacements, $\omega_1 > 0$ and $\omega_2 > 0$ are the natural frequencies (angular velocity), and $K_1 \geq 0$ and $K_2 \geq 0$ are the coupling constants. This system tends to synchronise in phase. Think of the two joggers analogy presented above and shown in Fig. 1. If jogger θ_1 is running faster than jogger θ_2, then term $K_1 \sin(\theta_2 - \theta_1)$ in the first equation becomes negative and thus reduces speed ω_1 of jogger θ_1. Synchronisation is also true for the reversed scenario where slower running jogger will gain speed. Let's study solution trajectories of Sys. (2) on the surface of a torus.

If $K_1 = K_2 = 0$, then Sys. (2) takes the form

$$\begin{aligned} \dot{\theta}_1 &= \omega_1, \\ \dot{\theta}_2 &= \omega_2, \end{aligned} \tag{3}$$

and we are left with a decoupled system. The solution to Sys. (3) is obtained by integrating the relevant equations

$$\begin{aligned} \int \dot{\theta}_1 \mathrm{d}t &= \int \omega_1 \mathrm{d}t, \\ \int \dot{\theta}_2 \mathrm{d}t &= \int \omega_2 \mathrm{d}t, \end{aligned} \tag{4}$$

since ω_1 and ω_2 are constant

$$\begin{aligned} \theta_1(t) &= \omega_1 t + C_1, \\ \theta_2(t) &= \omega_2 t + C_2. \end{aligned} \tag{5}$$

The integration constants can be resolved on the boundary $C_1 = C_2 = 0$. The solution trajectories on the 2π-periodic square for any initial condition are straight lines with constant slope $\mathrm{d}\theta_2/\mathrm{d}\theta_1 = \omega_2/\omega_1$. There are two qualitatively different cases, depending on whether the slope is a **rational** or an **irrational** number.

1.2.1 Periodic solution for rational slope

If the slope is **rational**, then $\omega_1/\omega_2 = p/q \in \mathbb{Q}$ for some integers $p, q \in \mathbb{Z}$ with no common factors. In this case all trajectories are closed orbits on the torus, because θ_1 completes p revolutions in the same time that θ_1 completes q revolutions.

For example, Fig. 3 shows a trajectory on a 2π-periodic square with $p = 3$ and $q = 2$. When plotted on a torus (in three dimensions), the same trajectory gives a **trefoil knot**. All **co-prime** p, q pairs produce knotted trajectories called the **toroidal knots**. Two integers p and q are said to be relatively prime, mutually prime, or co-prime if the only positive integer (factor) that divides both of them is 1. The closed knotted trajectory for $p = 5$ and $q = 2$ is called the **cinquefoil knot** (also called the pentafoil knot, the surgeon's knot or the Solomon's seal knot). Slide 4 shows a trefoil knot and a cinquefoil knot as they appear on a torus. We get a closed spiralling trajectory similar to a closed spring, not a knot, for non co-prime p and q, e.g., $p = 10$, $q = 2$.

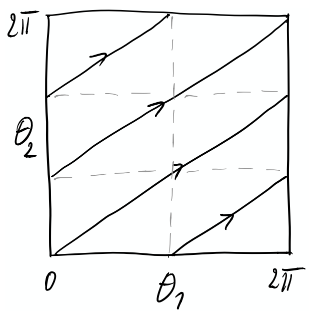


Figure 3: Trefoil knot where $p = 3$ and $q = 2$ shown on the 2π-periodic square.

The following numerical file visualises closed trajectories as they appear on the surface of a torus for rational slopes ω_1/ω_2.

Numerics: nb#1

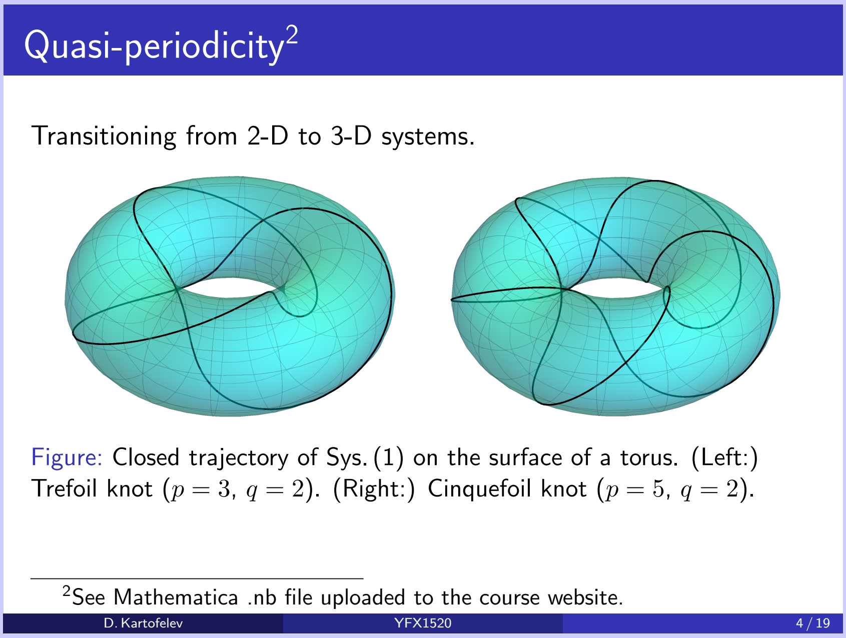
Trajectories on the surface of a torus: interactive 3-D plot. Periodic and quasi-periodic trajectories on the surface of a torus. Trefoil (for $p = 3$, $q = 2$) and cinquefoil knots (for $p = 5$, $q = 2$).

The following numerical file shows numerical solution to decoupled Sys. (3) and its phase portrait.

Numerics: nb#2

Quasi-periodic oscillators. Quasi-periodic decoupled system: periodic solution, quasi- periodic solution. The Fourier and power spectra of the solutions.

SLIDE: 4



1.2.2 Quasi-periodic solution for irrational slope

If the slope is **irrational** $\omega_1/\omega_2 \in \mathbb{P}$, then the solution is said to be **quasi-periodic**. In this case trajectories will never close into themselves for $t \to \infty$. This implies that a time-domain solution will never repeat itself. Figure 4 shows this scenario. How can we be sure that trajectories will never close? Any closed trajectory necessarily makes an integer number of revolutions in both θ_1 and θ_2; hence the slope would have to be rational, contrary to assumption. Furthermore, when the slope is irrational, each trajectory is **dense** for $t \gg 1$ on the torus: in other words, each trajectory comes arbitrarily close to any given point on the torus. This is not to say that the trajectory passes through each point; it just comes arbitrarily close.

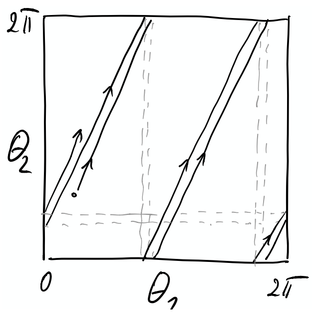


Figure 4: A trajectory with irrational slope shown on the 2π-periodic square. The trajectory is not closing into itself for $t \to \infty$.

Quasi-periodicity is significant because it is a new type of long-term behaviour. Unlike the earlier entries: fixed point, closed orbit, limit-cycles, homoclinic and heteroclinic orbits; quasi-periodicity occurs only on a torus—a three-dimensional object.

The following numerical files shows the numerical solution to decoupled Sys. (3) with its phase portrait and the dense three-dimensional trajectory of the quasi-periodic solution for irrational slope ω_1/ω_2.

NUMERICS: NB#2

Quasi-periodic oscillators. Quasi-periodic decoupled system: periodic solution, quasi- periodic solution. The Fourier and power spectra of the solutions.

NUMERICS: NB#1

Trajectories on the surface of a torus: interactive 3-D plot. Periodic and quasi-periodic trajectories on the surface of a torus. Trefoil (for $p = 3$, $q = 2$) and cinquefoil knots (for $p = 5$, $q = 2$).

The dense trajectory on the surface of a torus corresponding to the trajectory shown in Fig. 4.

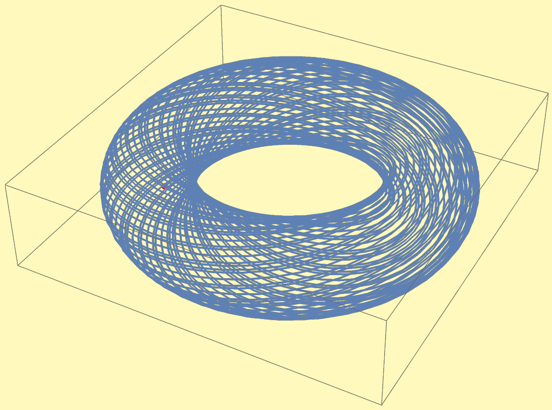


Figure 5: Dense trajectory on the surface of a torus for irrational slope $\omega_1/\omega_2 \in \mathbb{P}$.

Note: Quasi-periodicity is not *chaotic*. In a quasi-periodic system two close-by initial conditions are **linearly diverging** with the passage of time, see Fig. 6. **In the case of *chaos* the divergence must be exponential.** This notion will be further developed in future lectures.

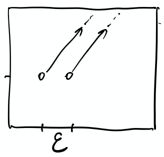


Figure 6: 2π-periodic square showing two close-by initial conditions, where distance $\varepsilon \ll 1$.

1.3 Example: Coupled case $K_1, K_2 \neq 0$

Now we consider the coupled case of Sys. (2) given in the form

$$\begin{cases} \dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1), \\ \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2), \end{cases}$$

where $\omega_1, \omega_2 > 0$ and $K_1, K_2 > 0$. The dynamics can be deciphered by looking at the phase difference $\phi = \theta_1 - \theta_2$. Then the temporal dynamics of Sys. (2) yields

$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 + K_1 \sin(\theta_2 - \theta_1) - \omega_2 - K_2 \sin(\theta_1 - \theta_2), \tag{6}$$

$$\dot{\phi} = \omega_1 - K_1 \sin(\theta_1 - \theta_2) - \omega_2 - K_2 \sin(\theta_1 - \theta_2), \tag{7}$$

$$\dot{\phi} = \omega_1 - K_1 \sin\phi - \omega_2 - K_2 \sin\phi, \tag{8}$$

$$\dot{\phi} = \omega_1 - \omega_2 - (K_1 + K_2) \sin\phi. \tag{9}$$

The dynamics depends on the interplay between $|\omega_1 - \omega_2|$ and $K_1 + K_2$ (*cf.* Lecture 7: Analysis of SNIPER bifurcation).

1.3.1 Periodic phase-locked solution, $|\omega_1 - \omega_2| < K_1 + K_2$

The dynamics of phase difference (9) is studied on the one-dimensional phase portrait shown in Fig. 7. We see that there are two fixed point for $\phi \in [0, 2\pi]$. These fixed points are implicitly defined by

$$\dot{\phi} = 0 \quad \Rightarrow \quad \omega_1 - \omega_2 - (K_1 + K_2) \sin\phi^* = 0 \quad \Rightarrow \quad \sin\phi^* = \frac{\omega_1 - \omega_2}{K_1 + K_2}. \tag{10}$$

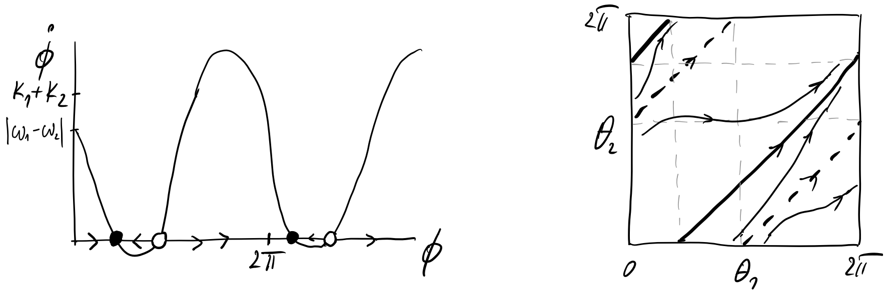


Figure 7: (Left) One-dimensional phase portrait of phase difference ϕ where all trajectories are approaching the stable fixed point. (Right) 2π-periodic square showing stable (solid bold line) and unstable (dashed bold line) non-isolated fixed points or trajectories on a torus corresponding to fixed points (10) shown on the left. The slope of stable and unstable locked solutions is 1, since $\omega^* = \dot\theta_1 = \dot\theta_2$.

As Fig. 7 shows, all trajectories of Sys. (9) approach asymptotically the stable fixed point. Therefore, back on the torus, the trajectories of Sys. (2) approach a stable **phase-locked** solution in which the oscillators are separated by a constant phase difference ϕ^*. The phase-locked solution is periodic for $t \gg 1$; in fact, both oscillators run at the constant frequency given by $\omega^* = \dot\theta_1 = \dot\theta_2 = \omega_2 + K_2 \sin\phi^*$. Substituting for $\sin\phi^*$, using (10), yields

$$\omega^* = \omega_2 + K_2 \frac{\omega_1 - \omega_2}{K_1 + K_2} \quad \Rightarrow \quad \omega^* = \frac{K_1\omega_2 - K_2\omega_1}{K_1 + K_2}. \tag{11}$$

This frequency is called the **compromise frequency** because it lies between the natural frequencies of the two oscillators. The compromise is not generally halfway; instead the frequencies are shifted by an amount proportional to the coupling strengths K_1 and K_2, as shown by identity (11). From three-dimensional point of view stable trajectory ω^* acts like a **3-D limit-cycle** for all initial conditions starting on the surface of this torus. The following numerical file shows numerical solution to coupled Sys. (2) and its phase portrait.

NUMERICS: NB#3
Quasi-periodic coupled oscillations: the Fourier and power spectra of the solutions.

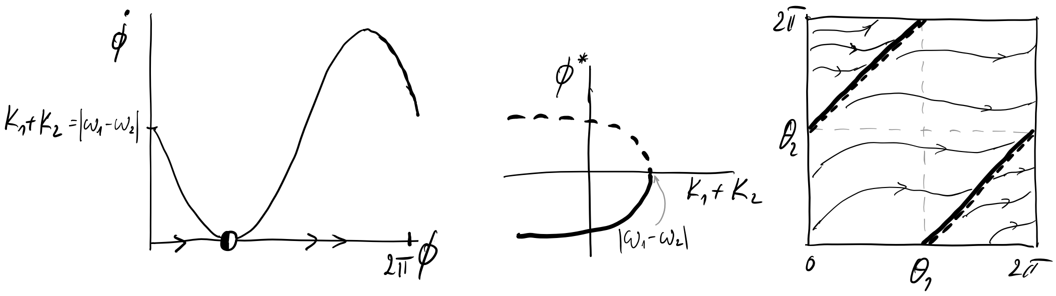


Figure 8: (Left) One-dimensional phase portrait of phase difference ϕ featuring half-stable fixed point. (Middle) Bifurcation diagram where bifurcation point $(K_1 + K_2)^* = |\omega_1 - \omega_2|$. (Right) 2π-periodic square showing half-stable (bold line) non-isolated fixed point or trajectory on a torus corresponding to the fixed points shown on the left.

1.3.2 Quasi-periodic solution with half-stable fixed point, $|\omega_1 - \omega_2| = K_1 + K_2$

If we pull the natural frequencies apart, say by detuning one of the oscillators, then the locked solutions approach each other and coalesce/merge when $|\omega_1 - \omega_2| = K_1 + K_2$, see Fig. 8. Thus the locked solution is destroyed in a **saddle-node coalescence of cycles bifurcation**. Figure 8 (Left) shows the one-dimensional phase portrait of phase difference ϕ for the current case. Figure 8 (Middle) shows the saddle-node bifurcation diagram which corresponds to the saddle-node coalescence of cycles bifurcation taking place on the torus. The following numerical file shows numerical solution to coupled Sys. (2) and its phase portrait.

> NUMERICS: NB#3
>
> Quasi-periodic coupled oscillations: the Fourier and power spectra of the solutions.

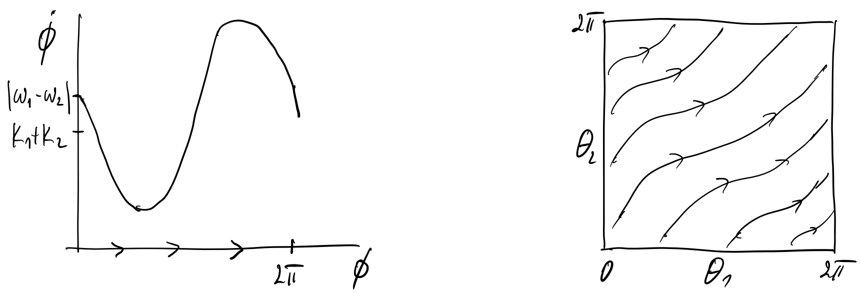


Figure 9: (Left) One-dimensional phase portrait of phase difference ϕ. (Right) Flow on the 2π-periodic square corresponding to the phase portrait shown on the left.

1.3.3 Quasi-periodic or periodic solution, $|\omega_1 - \omega_2| > K_1 + K_2$

After the bifurcation, the flow is like that in the decoupled case studied earlier: we have either a quasi-periodic for $\omega_1/\omega_2 \in \mathbb{P}$ or a rational periodic flow for $\omega_1/\omega_2 \in \mathbb{Q}$. The only difference is that now the trajectories on the 2π-periodic square are curvy, not straight. Figure 9 shows the dynamics corresponding to this case. The following numerical file shows numerical solution to coupled Sys. (2) and its phase portrait.

> NUMERICS: NB#3
>
> Quasi-periodic coupled oscillations: the Fourier and power spectra of the solutions.

An interactive overview notebook file for all possible behaviours of Sys. (2) is linked below.

> NUMERICS: NB#4
>
> Quasi-periodic coupled oscillations: interactive code.

2 3-D systems and introduction to chaos

The general form of three-dimensional or third order homogeneous systems is the following

$$\begin{cases} \dot{x} = f_1(x, y, z), \\ \dot{y} = f_2(x, y, z), \\ \dot{z} = f_3(x, y, z), \end{cases} \tag{12}$$

where f_1, f_2, and f_3 are the given functions. Fixed point/s (x^*, y^*, z^*) of the above system is/are defined by

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \\ \dot{z} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} f_1(x^*, y^*, z^*) = 0, \\ f_2(x^*, y^*, z^*) = 0, \\ f_3(x^*, y^*, z^*) = 0. \end{cases} \tag{13}$$

Phase portrait and solution trajectories of a 3-D system are now visualised in three dimensions, see Sec. 3 and Slides 16 and 17.

2.1 Quasi-periodic solution in 3-D systems

Note: If you see trajectories of a three-dimensional system ending up on the surface of a torus, then there is a possibility of quasi-periodic solution existing for that system.

2.2 Definition of chaos

There is no rigorous mathematical definition of chaos. This shouldn't defer us from trying. In the next week's lecture we will give a more detailed *conceptual* definition of chaos in addition to the information shown below.

SLIDES: 5–8

What happens if you google "chaos"?

Chaos, dictionary definition

Chaos[3] in day-to-day laymen jargon (colloquial meaning):

- a state of utter confusion or disorder.
- a total lack of organisation or order.
- complete confusion and disorder; *a state in which behaviour and events are not controlled by anything.*
- a state of things in which chance is supreme; *especially, the confused unorganised state of primordial matter before the creation of distinct forms.*
- any confused, disorderly mass: *a chaos of meaningless phrases.*

[3] Source: various online dictionaries.

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Chaos in mathematics and physics

Chaos theory is the field of study in mathematics that studies the behaviour of dynamical systems that are *highly* sensitive to initial conditions – a response popularly referred to as the "**butterfly effect**". Small differences in initial conditions (such as those due to rounding errors in numerical computation or measurement uncertainty) yield widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general. This happens even though these systems are *deterministic*, meaning that their future behaviour is fully determined by their initial conditions, with no random (stochastic) elements involved. In other words, the *deterministic nature* of these systems does not make them *predictable*. This behaviour is known as **deterministic chaos**, or simply **chaos**. Chaotic behaviour exists in many natural systems, such as weather and climate. It also occurs spontaneously in some systems with artificial components, such as road traffic.

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The colloquial meaning of the term "chaos" is often incorrectly associated with the mathematical use of the term. These terms have *nothing* in common.

Chaos in mathematics and physics

The fact that deterministic system is not predictable (determined) in *practice* is not an internally contradicting statement, its a manifestation of a **new mathematical property or type of solution** of higher order (order more than two) nonlinear systems, called **chaos**. Also, this long-term **aperiodic** solution is qualitatively different from the periodic and quasi-periodic solutions since solutions with slightly different initial conditions deviate exponentially.

The **chaos** was summarised by Edward Lorenz as:
Chaos – when the present determines the future, but the approximate present does not approximately determine the future.

When predicting *distant* future states of a chaotic system one can never know the starting point accurately enough.

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Chaos in mathematics and physics

The **chaos theory** explains deterministic systems which in principle can be predicted, for a time, then appear to become random. The amount of time patterns can be predicted depends on a time scale (the Lyapunov time) determined by the system's dynamics.

Chaos is a property of reality that we sense when we try to predict *distant* future.

SRB measure (Sinai-Ruelle-Bowen measure) – If statistics of trajectories of a system are **insensitive** to initial conditions or small differences of initial conditions then we say that the system has a SRB measure.

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The confusing terms "chaos" and "chaos *theory*" are not useful nor even needed, since as mentioned above, there are no agreed upon definition of chaos. All aspects of nonlinear and *chaotic* dynamical systems studied in future lectures can be characterised without using the terms "chaos" or "chaotic"—we have more accurately descriptive and/or precise terminology. In addition, all essential results of chaos theory are derived by and directly borrowed from other well-established science disciplines.

2.3 The Lorenz mill

We begin our study of chaos with a classical example—the **Lorenz mill**. The simplest version is a toy waterwheel with leaky paper cups suspended from its rim. Water is poured in steadily from the top. If the flow rate is too slow, the top cups never fill up enough to overcome friction, so the wheel remains motionless. For faster inflow, the top cup gets heavy enough to start the wheel turning. Eventually the wheel settles into a steady rotation in one direction or the other. By symmetry, rotation in either direction is equally possible; the outcome depends on the initial conditions.

By increasing the flow rate still further, we can destabilise the steady rotation. Then the motion becomes *chaotic*: the wheel rotates one way for a few turns, then some of the cups get too full and the wheel doesn't have enough inertia to carry them over the top, so the wheel slows down and may even reverse its direction. Then it spins the other way for a while. The wheel keeps changing direction **erratically**. Equations of motion of the Lorenz mill are shown below.

SLIDES: 9–11

Chaotic systems: The Lorenz mill[4]

Equations of motion of the **Lorenz mill** or chaotic water wheel are

$$\begin{cases} \dot{a}_1 = \omega b_1 - K a_1, \\ \dot{b}_1 = -\omega a_1 - K b_1 + q_1, \\ \dot{\omega} = -\dfrac{\nu}{I}\omega + \dfrac{\pi G r}{I} a_1, \end{cases} \tag{2}$$

where I is the moment of inertia, θ is the angle of the wheel, ω is the angular velocity, K is the liquid's leakage rate, ν is the damping rate, r is the radius of the wheel, G is the effective gravity constant. a_1 and b_1 are the Fourier amplitudes of the first modes of the liquid's mass distribution function

$$m(\theta, t) = \sum_{n=0}^{\infty} \left[a_n(t) \sin n\theta + b_n(t) \cos n\theta\right]. \tag{3}$$

[4] See Mathematica .nb file uploaded to the course website.

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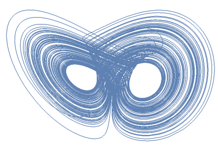
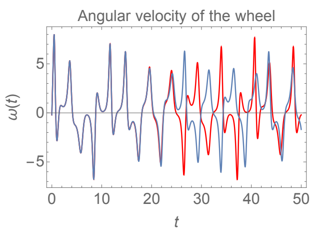
The derivation of the equations of motion for this problem is discussed in Chapter 9 of our main textbook.

Chaotic systems: The Lorenz mill

g_1 is the Fourier amplitude of the first mode of the liquid inflow mass distribution function

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta. \tag{4}$$





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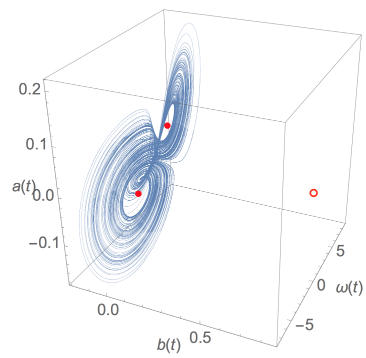
Chaotic systems: The Lorenz mill[5]



[5] See Mathematica .nb file uploaded to the course website.

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The Lorenz mill dynamics shows **strong dependence on a selection of initial conditions**.

Numerical file used to calculate the above results is linked below.

NUMERICS: NB#5

An example of a chaotic system: the Lorenz mill. Power spectra of the time-series solutions of the Lorenz mill system. Flow trajectory plotting in 3-D.

Includes a demonstration of the system's sensitive dependence on initial conditions.

Following slides show two video animations of the Lorenz mill and its dynamics.

SLIDES: 12, 13

Chaotic systems: The Lorenz mill and chaos

No embedded video files in this pdf

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Chaotic systems: The Lorenz mill, SRB measure

No embedded video files in this pdf

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Some statistical aspects of the rotation might not be sensitive to initial conditions (SRB measure).

2.4 The Lorenz attractor

SLIDES: 14, 15

Eduard N. Lorenz derived this three-dimensional system from a drastically simplified model of convection rolls in the atmosphere. The same equations also arise in simplified models of lasers, dynamos, thermosyphons, brushless direct current (DC) motors, electric circuits, chemical reactions, forward osmosis, and they exactly describe the motion of the Lorenz mill. The Lorenz mill is a specific case of the Lorenz attractor. The proof can be found in our main textbook.

Chaotic systems: The Lorenz attractor

The **Lorenz attractor:**[6] It can be shown that Sys. (2) is a specific case of a more general system in the form

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{5}$$

where σ, r, $b > 0$ are the control parameters.

Read: E. N. Lorenz, “Deterministic nonperiodic flow”. *Journal of the Atmospheric Sciences*, **20**(2), pp. 130–141 (1963). `http://dx.doi.org/10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2`

[6]See Mathematica .nb file uploaded to the course website.

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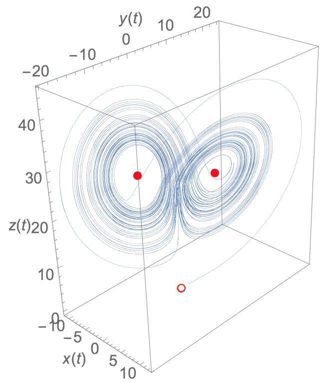
Chaotic systems: The Lorenz attractor



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The system has only two nonlinearities, the quadratic terms xy and xz, rendering it a relatively simple chaotic system, algebraically speaking.

Numerical file linked below contains the time-domain solution of the Lorenz attractor.

NUMERICS: NB#6

An example of a chaotic system: the Lorenz attractor. Numerical integration of the Lorenz attractor (interactive 3-D plot, interactive 2-D (x-z projection) plot).

Includes a dynamic simulation of the Lorenz flow.

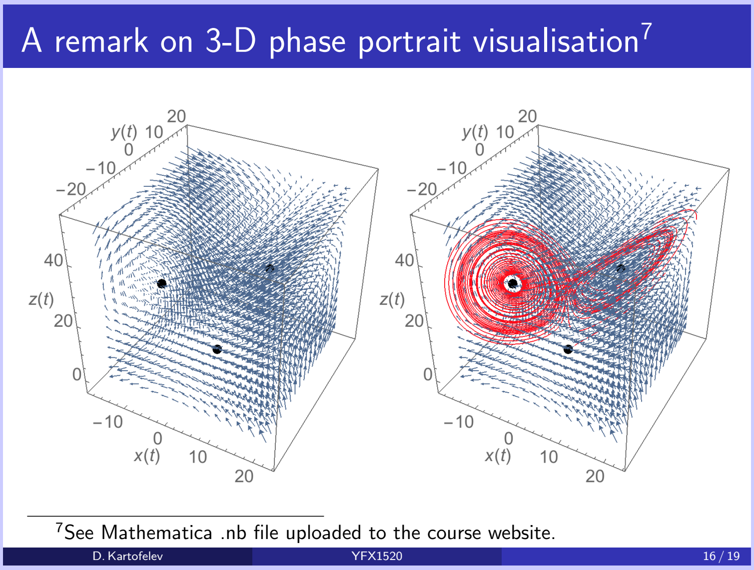
Reading suggestion

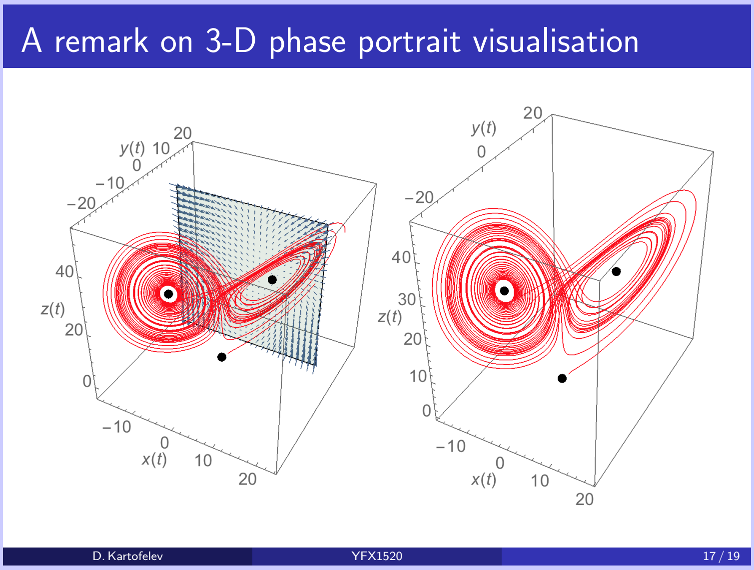
Eduard N. Lorenz’s original paper is deep, prescient, and surprisingly readable—look it up! Edward N. Lorenz (1917–2008) is often considered to be the discoverer of chaos, this is not true. Chaos was discovered by Japanese Professor of Electrical Engineering Yoshisuke Ueda (1936-).

Link	**File name**	**Citation**
Paper#1	paper1.pdf	Edward N. Lorenz, "Deterministic nonperiodic flow," *Journal of the Atmospheric Sciences*, **20**(2), pp. 130–141, (1963). doi:10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2

3 A remark on plotting 3-D phase portraits

Slides: 16, 17





Three-dimensional phase portraits are hard to visualise/read using vectors placed into three-dimensional projection. Slide 16 shows the vector field of the Lorenz attractor in the manner we have been doing it so far for two-dimensional phase portraits, with exception of raising dimensionality by one. A good graphical overview can be given by showing a single or couple trajectories as shown on Slide 17. This is especially true in the case of *chaotic* attractors explained/defined in future lectures.

Numerical file used to calculate the above results is linked below.

Numerics: nb#7

A remark on 3-D phase portrait plotting. 3-D phase portrait and flow visualisation of the Lorenz attractor.

4 Coursework

The coursework variant assigned to you is announced on the course webpage (or TalTech Moodle). A positively graded coursework is a **prerequisite for taking the exam**.

4.1 *Coursework requirements*

The coursework consists of two parts. The first part requires you to analyse a 2-D system and the second part requires the analysis of a 3-D system. Completed coursework may be handed over in two parts at any time during the semester.

Part 1: Analysis of a 2-D system

1. Perform linear analysis:
   - Find a fixed point or points of your system.
   - Linearise your system about the fixed point or points.
   - Plot the linearised phase portrait (suggestion).
   - Determine the type of the linearised fixed point or points.
   - Determine if/how changes in control parameter values influence the dynamics (type of fixed point/s) of your system.
2. Perform nonlinear analysis of the full homogeneous system using a computer:
   - Compare the type of the nonlinear fixed point or points with the corresponding linearised fixed point or points.
   - Plot the nonlinear phase portrait. Compare it with the linearised one.
   - Explain any discrepancy between the nonlinear and linearised systems if any occurs.
3. Perform nonlinear analysis of the non-homogeneous system (if applicable) using a computer:
   - Analyse the influence of the explicitly time dependant part of the system on the system dynamics.
   - Plot the nonlinear non-homogeneous phase portrait. Compare it with the homogeneous one.
   - Explain the obtained and presented results
4. Draw overall conclusions and comment on the presented results.

Part 2: Analysis of a 3-D system

1. Compare the known properties of strange attractors against your system.
2. Draw conclusions based on your analysis results.

How?

Above problems and specific tasks must be tackled with the use of analysis methods presented and discussed during the lectures.

Personal consultation

Before submitting the completed and finalised coursework for an evaluation, You have the right to consult the Lecturer and discuss your progress to ensure that the final submitted coursework is laking any technical mistakes and/or errors. Please do not hesitate to use this opportunity.

4.2 Coursework analysis tools

Students who lack coding skills are welcomed to use the numerical analysis tools provided by the Lecturer and others. These tools don't require any coding skills—just type in your system and study the results displayed. For additional information visit the course webpage. Below are linked the numerical and symbolic analysis tools created specially for this course.

NUMERICS: NB#8

Coursework analysis tools: Mathematica notebook for analysing 2-D and 3-D systems. Use of the tool does not require coding skills.

This tool is capable: performing linear and eigenanalysis of linear and nonlinear systems; to plot 2-D and 3-D phase portraits; and find/plot numerically integrated time-domain solutions of given systems.

Opening .nb files requires Wolfram Mathematica installation. Wolfram Player can't handle them. No coding skills required.

Revision questions

1. What is quasi-periodicity?
2. Can quasi-periodic system generate a chaotic solution? Why?
3. Do limit-cycles exist in 3-D phase spaces? Sketch an example.
4. What are 3-D and higher order systems?
5. What is chaos in the context of dynamical systems (deterministic chaos, chaos theory)?
6. Name properties of chaotic systems.
7. What does it mean that a chaotic system has a SRB measure (Sinai-Ruelle-Bowen measure)?
8. What is chaotic water wheel?
9. What is the Lorenz attractor?