Lecture 5: 2-D homogeneous nonlinear systems, linearisation of 2-D systems about its fixed points, stability and type of nonlinear fixed points, system's Jacobian matrix, homoclinic orbit, stable and unstable manifolds, conservative systems

Contents

1 Nonlinear 2-D systems

The system

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y), \end{cases} \tag{1}$$

is nonlinear for given **nonlinear functions** f and g. Fixed point (x^*, y^*) of the system is found by solving

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} f(x^*, y^*) = 0, \\ g(x^*, y^*) = 0, \end{cases} \tag{2}$$

for x^* and y^* simultaneously.

What can be said about the type and stability of a nonlinear fixed point. In this lecture we extend the linearisation technique developed for 1-D systems during Lecture 2. The hope is that we can approximate phase portraits near fixed points by that of a corresponding linear systems.

2 Linearisation of 2-D systems

2.1 Linearisation about a fixed point

Slides: 3–6

Linearisation of 2-D systems

Nonlinear system is given by

$$\begin{cases} \dot{x} = f(x,y), \\ \dot{y} = g(x,y). \end{cases} \tag{1}$$

Let's consider small perturbations/deviations: $|u| \ll 1$ in the x-direction, and $|v| \ll 1$ in the y-direction. Perturbed dynamics of the solution of Sys. (1) in close proximity to fixed point (x^*, y^*) thus is

$$\begin{cases} x(t) = x^* + u(t), \\ y(t) = y^* + v(t), \end{cases} \tag{2}$$

equivalently we write

$$\begin{cases} u(t) = x(t) - x^*, \\ v(t) = y(t) - y^*. \end{cases} \tag{3}$$

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Linearisation of 2-D systems

Temporal dynamics of perturbations u and v is the following:

$$\begin{cases} \dot{u} = (x - x^*)^{\cdot} = \dot{x} = f(x^* + u, y^* + v) = \\ \quad = f(x^*, y^*) + u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,y^*)} + O(u^2, v^2, uv) \approx \\ \quad \approx u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,y^*)} \\ \dot{v} = (y - y^*)^{\cdot} = \dot{y} = g(x^* + u, y^* + v) = \\ \quad = g(x^*, y^*) + u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,y^*)} + O(u^2, v^2, uv) \approx \\ \quad \approx u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,y^*)} \end{cases} \tag{4}$$

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Linearisation of 2-D systems

For a better overview we collect the above results:

$$\begin{cases} \dot{u} = u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,y^*)}, \\ \dot{v} = u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,y^*)}. \end{cases} \tag{5}$$

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Linearisation of 2-D systems

The matrix form for $\vec{u} = (u, v)^T$ is the following:

$$\dot{\vec{u}} = \left.\begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}\right|_{(x^*,y^*)} \cdot \vec{u} \equiv J\big|_{(x^*,y^*)} \cdot \vec{u}, \tag{6}$$

where matrix J is the Jacobian matrix of the given system. Neglecting higher order terms (h.o.t.) $O(u^2, v^2, uv)$ yields the linearisation about fixed point (x^*, y^*) in form (6).

Note: Higher order terms of order $O(uv)$ are also negligibly small since $|u|, |v| \ll 1$.

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Let's consider small perturbations/deviations $|u| \ll 1$ in the x-direction and $|v| \ll 1$ in the y-direction. Perturbed dynamics of the solution to Sys. (1) in close proximity to fixed point (x^*, y^*) thus is

$$\begin{cases} x(t) = x^* + u(t), \\ y(t) = y^* + v(t), \end{cases} \tag{3}$$

equivalently we write

$$\begin{cases} u(t) = x(t) - x^*, \\ v(t) = y(t) - y^*. \end{cases} \tag{4}$$

Temporal dynamics of small perturbations u and v in x and y directions of the phase portrait are

$$\begin{cases} \dot{u} = (x - \underbrace{x^*}_{\text{const.}})^{\cdot} = \dot{x} = f(x,y) = \begin{bmatrix}\text{See}\\ \text{Ex. (3)}\end{bmatrix} = f(x^* + u, y^* + v) = \begin{bmatrix}\text{Multivariable}\\ \text{Taylor}\\ \text{series about}\\ \text{f.p. } (x^*, y^*)\end{bmatrix} \\ = \overbrace{f(x^*, y^*)}^{=0} + (\cancel{x^*} + u - \cancel{x^*}) \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,y^*)} + (\cancel{y^*} + v - \cancel{y^*}) \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,y^*)} + \underbrace{O(u^2, v^2, uv)}_{\text{h.o.t.}} \\ \approx u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,y^*)}, \\ \dot{v} = (y - \underbrace{y^*}_{\text{const.}})^{\cdot} = \dot{y} = g(x,y) = \begin{bmatrix}\text{See}\\ \text{Ex. (3)}\end{bmatrix} = g(x^* + u, y^* + v) = \begin{bmatrix}\text{Multivariable}\\ \text{Taylor}\\ \text{series about}\\ \text{f.p. } (x^*, y^*)\end{bmatrix} \\ = \overbrace{g(x^*, y^*)}^{=0} + (\cancel{x^*} + u - \cancel{x^*}) \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,y^*)} + (\cancel{y^*} + v - \cancel{y^*}) \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,y^*)} + \underbrace{O(u^2, v^2, uv)}_{\text{h.o.t.}} \\ \approx u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,y^*)}. \end{cases} \tag{5}$$

For a better overview we collect the above results

$$\begin{cases} \dot{u} = u \left.\dfrac{\partial f}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial f}{\partial y}\right|_{(x^*,y^*)}, \\ \dot{v} = u \left.\dfrac{\partial g}{\partial x}\right|_{(x^*,y^*)} + v \left.\dfrac{\partial g}{\partial y}\right|_{(x^*,y^*)}. \end{cases} \tag{6}$$

The matrix form for $\vec{u} = (u, v)^T$ is the following:

$$\dot{\vec{u}} = \left.\begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}\right|_{(x^*,y^*)} \cdot \vec{u} \equiv J\big|_{(x^*,y^*)} \cdot \vec{u}, \tag{7}$$

where matrix J is the Jacobian matrix. Neglecting higher order terms (h.o.t.) $O(u^2, v^2, uv)$ yields the linearisation about fixed point (x^*, y^*) in form (7). Higher order terms of order $O(uv)$ are also negligibly small since $|u|, |v| \ll 1$. Typically, the <u>resulting system is qualitatively similar to its nonlinear counterpart near its fixed point, but not alway</u>!

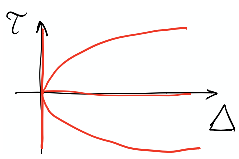

Figure 1: The τ vs. Δ classification graph showing the <u>borderline cases</u> that are shown with the red colour.

Note 1: Sometimes linearisation of a system changes the type of its fixed points. This is true for the **borderline cases** shown on the τ vs. Δ fixed point classification graph introduced in the previous lecture. Figure 1 shows these ambiguous borderline cases. The nonlinear terms of original systems can tip a borderline case to a nearby case in the τ-Δ plane.

In terms of the detailed flowchart, also presented during the previous lecture, these exceptions are:

- if $\Delta < 0$:
  Isolated fixed point
  CASE 1: **Saddle point**[1]
- if $\Delta = 0$:
  Non-isolated fixed points
  - if $\tau < 0$:
    CASE 5a: **Line of stable fixed points**[2]
  - if $\tau = 0$:
    CASE 5b: **Plane of fixed points**[3]
  - if $\tau > 0$:
    CASE 5a: **Line of unstable fixed points**[4]
- if $\Delta > 0$:
  Isolated fixed point
  - if $\tau < -\sqrt{4\Delta}$:
    CASE 2a: **Stable node**[5]
  - if $\tau = -\sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
      CASE 4a: **Stable degenerate node**[6]
    - if there are no uniquely determined eigenvectors (both are non-unique):
      CASE 4b: **Stable star**[7]
  - if $-\sqrt{4\Delta} < \tau < 0$:
    CASE 2b: **Stable spiral**[8]
  - if $\tau = 0$:
    CASE 3: **Centre**[9]
  - if $0 < \tau < \sqrt{4\Delta}$:
    CASE 2b: **Unstable spiral**[10]
  - if $\tau = \sqrt{4\Delta}$:
    - if there is one uniquely determined eigenvector (the other is non-unique):
      CASE 4a: **Unstable degenerate node**[11]
    - if there are no uniquely determined eigenvectors (both are non-unique):
      CASE 4b: **Unstable star**[12]
  - if $\sqrt{4\Delta} < \tau$:
    CASE 2a: **Unstable node**[13]

General notes and the exceptions related to the borderline cases shown in Fig. 1:

- For 2-D linear systems, the above predictions are always accurate.
- For 2-D nonlinear systems, when the above are used as predictions (see the superscripted numbering):
  - The descriptions are always correct for cases 1, 5, 8, 10, and 13 but can be inaccurate for cases 2, 3, 4, 6, 7, 9, 11, and 12.
  - Ambiguous cases 6, 7, 11, and 12 at least have their stability correctly determined.
  - If the system is conservative, a prediction of case 9 is accurate.

Note 2: In addition to the borderline cases mentioned above some nonlinear systems simply **don't have analogues** in the *world* of linear systems. For example there are systems with 2-D flow *indices* not equal to 1 or -1, which is the case with linear 2-D systems as they were introduced in previous lecture. The concept/notion of an index of a closed curve and the index theory are not discussed in this course.

2.2 Example: Ambiguous borderline case

Consider a system given in the following form:

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2), \\ \dot{y} = x + ay(x^2 + y^2), \end{cases} \tag{8}$$

where a is the control parameter. Fixed point $(x^*, y^*) = (0, 0)$.

2.2.1 Linearisation

We linearise Sys. (8) and perform the linear analysis. We need to find the Jacbian matrix and evaluate it at the fixed point $(x^*, y^*) = (0, 0)$.

$$J\big|_{(x^*,y^*)} = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix}\Bigg|_{(x^*,y^*)}, \tag{9}$$

where the matrix elements are the following:

$$\frac{\partial \dot{x}}{\partial x}\bigg|_{(x^*,y^*)} = \left(3ax^2 + ay^2\right)\big|_{(0,0)} = 0, \qquad \frac{\partial \dot{x}}{\partial y}\bigg|_{(x^*,y^*)} = (-1 + 2axy)\big|_{(0,0)} = -1, \tag{10}$$

$$\frac{\partial \dot{y}}{\partial x}\bigg|_{(x^*,y^*)} = (1 + 2axy)\big|_{(0,0)} = 1, \qquad \frac{\partial \dot{y}}{\partial y}\bigg|_{(x^*,y^*)} = \left(ax^2 + 3ay^2\right)\big|_{(0,0)} = 0. \tag{11}$$

The resulting evaluated Jacobian matrix thus is the following:

$$J\big|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

The linearised system, defined by (7), is given by

$$\dot{\vec{u}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u} \quad \Rightarrow \quad \begin{cases} \dot{u} = -v, \\ \dot{v} = u, \end{cases} \tag{13}$$

where $\vec{u} = (u, v)^T$. It is important to note that the linearised system does not depend on the control parameter a.

2.2.2 A second look at the linearised system

Let's think logically about the linearisation process. What does linearisation imply? Basically, we are eliminating nonlinear *parts*/terms of a system. For any system where the fixed point is at the origin $(x^*, y^*) = (0, 0)$ and where the linear and nonlinear terms are explicitly distinct as is the case here (8)

$$\begin{cases} \dot{x} = -y + \underbrace{ax(x^2 + y^2)}_{\text{nonlinear terms}}, \\ \dot{y} = x + \overbrace{ay(x^2 + y^2)}, \end{cases}$$

we may simply throw out the nonlinear terms and we are left with

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x. \end{cases} \tag{14}$$

Since (4) holds and $(x^*, y^*) = (0, 0)$, then $u = x$ and $v = y$ and we write

$$\begin{cases} \dot{u} = -v, \\ \dot{v} = u. \end{cases}$$

This system is the same as the one derived previously, see (13).

2.2.3 Linear analysis: Classification of the fixed point of the linearised system

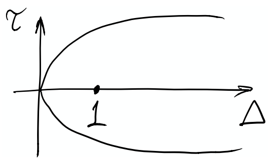

Figure 2: The τ vs. Δ classification graph showing the fixed point type as being the centre.

Trace τ of system matrix of Sys. (13) is

$$\tau = \operatorname{tr}\left(J\big|_{(x^*,y^*)}\right) = \operatorname{tr}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 + 0 = 0, \tag{15}$$

and determinant Δ is

$$\Delta = \det\left(J\big|_{(x^*,y^*)}\right) = \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (0 \cdot 0) - (-1 \cdot 1) = 1. \tag{16}$$

Figure 2 shows the fixed point position on the τ vs. Δ graph. The linear fixed point is a Lyapunov stable **centre**. Since linear fixed point is a centre (ambiguous borderline case) we have a reason to be cautious about the obtained result.

2.2.4 Nonlinear analysis: What is really going on?

Let's analyse nonlinear Sys. (8) to check if the linear analysis is accurate. The following analysis is based on the observation that nonlinear terms in Sys. (8) feature $x^2 + y^2$. Most likely they have something to do with circular flow and therefore we should consider polar coordinate representation of the system. Our aim is to simplify the system analytically and to analyse the dynamics of the resulting more intuitive system.

SLIDES: 7–12

Example: Ambiguous borderline case

An example where linear center is changed by nonlinearity (a borderline case).

Consider the following system:

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2), \\ \dot{y} = x + ay(x^2 + y^2), \end{cases} \tag{7}$$

where a is the control parameter[1].

[1] See Mathematica .nb file uploaded to the course webpage.

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Analysis of the nonlinear dynamics

Sys. (7) is analysed in polar coordinates. Usually a direct coordinate transform

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \tag{8}$$

where $r = r(t)$ and $\theta = \theta(t)$, is used. This approach may prove to be work-intense. Let's instead use a valid identity in the form:

$$\boxed{\begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1}\dfrac{y}{x}. \end{cases}} \tag{9}$$

We are searching a system in the form:

$$\begin{cases} \dot{r} = f(r, \theta), \\ \dot{\theta} = g(r, \theta), \end{cases} \tag{10}$$

where functions $f(r, \theta)$ and $g(r, \theta)$ are to be determined.

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Analysis of the nonlinear dynamics

Substituting (9) into original Sys. (7) results in

$$\begin{cases} \dot{x} = -y + ax(x^2+y^2) = -y + axr^2, \\ \dot{y} = x + ay(x^2+y^2) = x + ayr^2. \end{cases} \tag{11}$$

Using (9) we write

$$r^2 = x^2 + y^2, \tag{12}$$

where $x = x(t)$, $y = y(t)$ and $r = r(t)$. We are interested in temporal dynamics, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = \frac{\mathrm{d}}{\mathrm{d}t}(x^2+y^2), \tag{13}$$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} \quad | \div 2, \tag{14}$$

$$\boxed{r\dot{r} = x\dot{x} + y\dot{y}.} \tag{15}$$

This identity is used here in connection with Sys. (11) to derive the first equation of sought Sys. (10).

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Analysis of the nonlinear dynamics

Substituting (11) into the right-hand side of (15) results in

$$\begin{aligned} r\dot{r} &= x(-y + axr^2) + y(x + ayr^2) \\ &= -\cancel{xy} + ax^2r^2 + \cancel{xy} + ay^2r^2 \\ &= a\underbrace{(x^2+y^2)}_{r^2}r^2 = ar^4. \end{aligned} \tag{16}$$

Above result can be simplified:

$$r\dot{r} = ar^4 \quad | \div r, \tag{17}$$

$$\boxed{\dot{r} = ar^3.} \tag{18}$$

We have found the first equation of sought Sys. (10). We are one step closer to the polar representation of the original problem, given by Sys. (7).

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Analysis of the nonlinear dynamics

The second equation of sought Sys. (10) is found in the same way. Using (9) we write (temporal dynamics)

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = \frac{\mathrm{d}}{\mathrm{d}t}\left(\tan^{-1}\frac{y}{x}\right) \Rightarrow \begin{bmatrix} \theta = \theta(t), \\ x = x(t), \\ y = y(t), \\ \text{chain rule,} \\ \text{simplify} \end{bmatrix} \Rightarrow 1 \cdot \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{\underbrace{x^2+y^2}_{r^2}}. \tag{19}$$

Substituting (11) into the right-hand side of the obtained result gives us

$$\begin{aligned} \dot{\theta} &= \frac{x(x + ayr^2) - y(-y + axr^2)}{r^2} \\ &= \frac{x^2 + \cancel{axyr^2} + y^2 - \cancel{axyr^2}}{r^2} = \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1, \end{aligned} \tag{20}$$

$$\boxed{\dot{\theta} = 1.} \tag{21}$$

We have found the second equation of sought Sys. (10).

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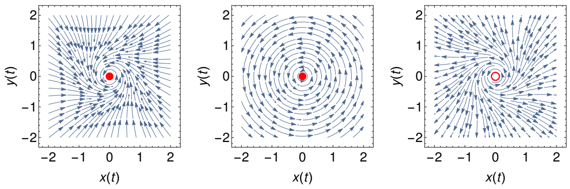
Analysis of the nonlinear dynamics

Sys. (7) has been represented in polar coordinates. Resulting <u>decoupled</u> equations (18) and (21) are

$$\begin{cases} \dot{x} = -y + ax(x^2+y^2) \\ \dot{y} = x + ay(x^2+y^2) \end{cases} \Rightarrow \quad \begin{aligned} \dot{r} &= ar^3, \\ \dot{\theta} &= 1. \end{aligned}$$



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Polar coordinate representation is intuitive and easy to grasp. Here, angular velocity $\dot{\theta}$ is constant and the changes in radial direction $\dot{r}$ are either growing <u>for $a > 1$ resulting in an **unstable spiral**</u> or diminishing <u>for $a < 1$ resulting in a **stable spiral**</u>. If $a = 0$, then there is no change in the radial direction and we have a **centre**—a set of closed orbits in the phase plane. Case $a = 0$ corresponds to a purely linear behaviour (nonlinear terms are rendered absent from Sys. (8) for $a = 0$) and agrees with the linearised Sys. (13). We conclude that <u>linearisation changed the type of the fixed point</u> and that the underlying cause was <u>nonlinearity</u>. The nonlinear terms tipped a borderline linear centre to a nearby spiral in the $\Delta\tau$-plane.

NUMERICS: NB#1

Effect of nonlinearity on a linear centre (a borderline case). Numerical solution and phase portrait of the same problem.

A lesson to learn is that centres and other **borderline cases are delicate** and they can be altered by nonlinearity. Figure 3 shown how a trajectory of a centre is destroyed by a nonlinear perturbation.

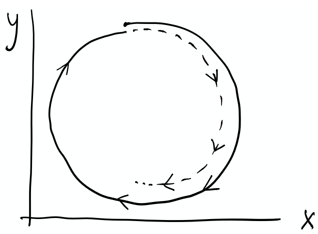


Figure 3: A trajectory of a centre being perturbed by nonlinearity. Nonlinearity forces the trajectory to miss its initial condition causing it to spiral inwards (by the properties of the underlying vector field).

3 The Lotka-Volterra models

3.1 Competitive cohabitation of rabbits and sheep

The model has the following form:

$$\begin{cases} \dot{x} = x(3-x) - 2xy, \\ \dot{y} = y(2-y) - xy, \end{cases} \tag{17}$$

where $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$ are the sizes of the rabbit and sheep populations, respectively. Both species are competing for the same food source—grass and other vegetation.

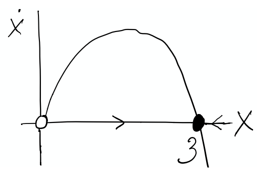


Figure 4: 1-D phase portrait of the logistic equation where carrying capacity $x^* = K = 3$.

The inner-workings of the model can be explained using the logistic equation presented in previous lectures. Let's focus only on one animal species. If we assume no change in sheep population size $\dot{y} = 0$ and no sheep to begin with $y(0) = 0$, then model (17) takes the following form:

$$\begin{cases} \dot{x} = x(3-x) \\ \dot{y} = 0 \end{cases} \quad \Rightarrow \quad \dot{x} = x(3-x). \tag{18}$$

Figure 4 shows the phase portrait of the resulting model equation. The eventual population size of the rabbits is determined only by the quantity of available food (carrying capacity $x^* = K = 3$).

If $y \neq 0$ then term $-2xy$ of Sys. (17), which is proportional to the sheep population size, will cause some additional restrictions on the rabbit population size. This term is the death rate associated with the existence of the competing species (sheep). The same logic applies to the second equation of Sys. (17).

Next, we perform linear analysis. Fixed points are found by solving

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x^*(3-x^*) - 2x^*y^* = 0, \\ y^*(2-y^*) - x^*y^* = 0, \end{cases} \tag{19}$$

for x^* and y^*. The fixed points are the following: $(x^*, y^*) = (0,0)$, $(0,2)$, $(3,0)$, $(1,1)$. Jacobian matrix of Sys. (17) is

$$J = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}. \tag{20}$$

The Jacobian matrices evaluated about fixed points (x^*, y^*) are the following:

$$J\big|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \tag{21}$$

here we have a diagonal matrix. This means that the eigenvalues are the diagonal elements. $\lambda_1 = 3$ and $\lambda_2 = 2$, and since $\lambda_1,\ \lambda_2 > 0 \in \mathbb{R}$ (distinct real eigenvalues with the same sign), then according to our classification graph (the Δ vs. τ plot) we have an **unstable node**;

$$J\big|_{(0,2)} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}, \tag{22}$$

here we have a triangular matrix. This means that the eigenvalues are the diagonal elements $\lambda_1 = -1$ and $\lambda_2 = -2$, and since $\lambda_1, \lambda_2 < 0 \in \mathbb{R}$ we have a **stable node**;

$$J\big|_{(3,0)} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}, \tag{23}$$

here once again we have a triangular matrix. Eigenvalue $\lambda_1 = -3$ and $\lambda_2 = -1$, and since $\lambda_1, \lambda_2 < 0 \in \mathbb{R}$ we have a **stable node**;

$$J\big|_{(1,1)} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}, \tag{24}$$

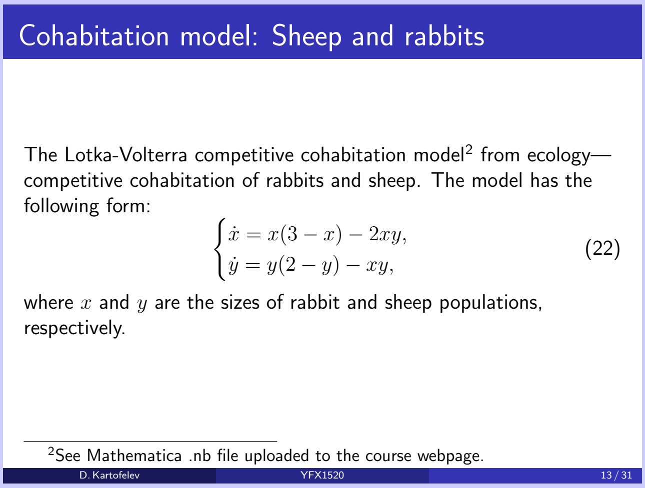
here $\Delta = (-1 \cdot -1) - (-2 \cdot -1) = 1 - 2 = -1$. Since $\Delta < 0$ we have a **saddle**. None of the above are the borderline cases this means that we can trust these results and proceed with the construction of the phase portrait. We construct the phase portrait by combining all of the above results into a single graph or by using a computer to do it for us.

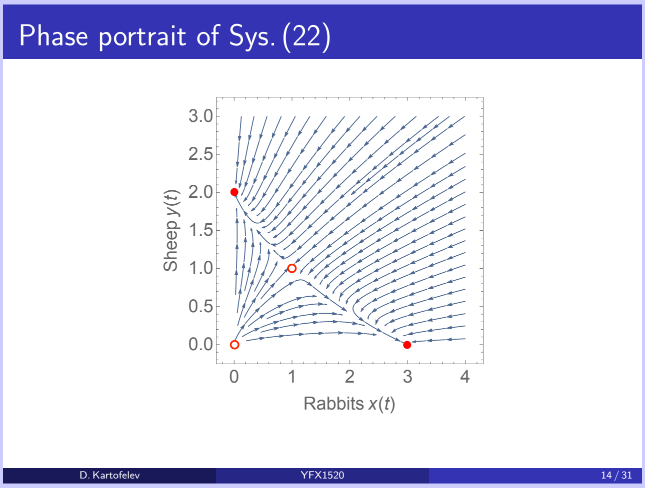
Numerics: NB#2

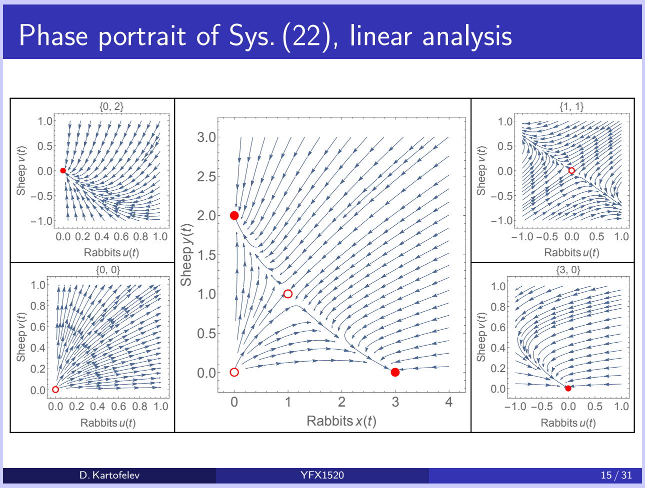
Competitive cohabitation of sheep and rabbits (the Lotka-Volterra predator–pray model). Linear analysis of fixed points and system manifolds. Numerical solution and phase portrait.

Numerical calculation of results (19)–(24) are also presented here.

Slides: 13–16







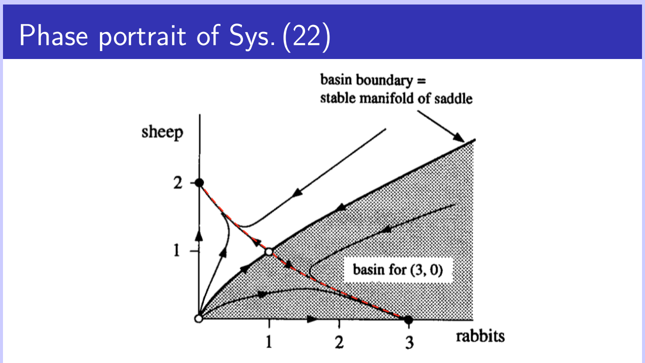


Figure: Phase portrait. The portrait features two basins of attraction corresponding to the stable fixed points. The stable manifold of the saddle is located at the basin boundaries. The unstable manifold is shown with the red dashed line.

The axes of the phase portrait are the **invariant sets**. If you start on an invariant set you will remain there for all time. Example: if $x = 0$ (vertical axis), then x will alway be 0. The rabbits do not just pop-out out of nothing.

The **basins of attraction** are separated by the **stable manifold** of the saddle point which is called the **basin boundary**. Biological interpretation of the phase portrait is the following: It shows that one species generally drives the other to extinction. Trajectories starting below the stable manifold lead to eventual extinction of the sheep, while those starting above lead to eventual extinction of the rabbits. This dichotomy is referred to as the **principle of competitive exclusion** which states that two species competing for the same limited resource typically cannot coexist.

The following numerical file contains the time-domain solutions corresponding to the phase portraits shown on Slides 14–16.

NUMERICS: NB#2

Competitive cohabitation of sheep and rabbits (the Lotka-Volterra predator–pray model). Linear analysis of fixed points and system manifolds. Numerical solution and phase portrait.

The phase portraits shown on the above lecture slides and the corresponding time-domain solutions.

3.2 A predator–prey model for fish and sharks

What happens when one species preys on another. When predator population size depends on population size of prey species and vice versa. Consider the following example.

SLIDES: 17, 18

Predator–prey model: Fish and sharks

Home assignment. Study the dynamics. How is this model different from the "sheep and rabbits" model?

Model is given in the following form:

$$\begin{cases} \dot{x} = \alpha x - \beta x y, \\ \dot{y} = \gamma \beta x y - \delta y, \end{cases} \tag{23}$$

where x is the number/concentration of the prey species, y is the number/concentration of the predator species, α is the prey species' population growth rate, β is the predation rate of y upon x, γ is the assimilation efficiency of y, and δ is the mortality rate of the predator species[3].

[3]See Mathematica .nb file uploaded to the course webpage.

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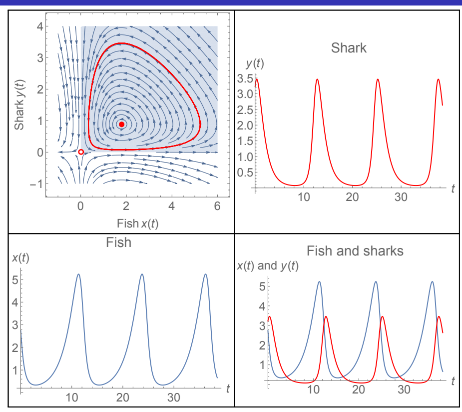
Predator–prey model: Fish and sharks



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In the model system, the predators thrive when there are plentiful prey but, ultimately, outstrip their food supply and decline. As the predator population is low, the prey population will increase again. These dynamics continue in a population cycle of growth and decline.

The following interactive numerical file contains the time-domain solutions showed on Slide 18.

NUMERICS: NB#3

The Lotka-Volterra predator–pray model for sharks and fish (cyclic behaviour). Numerical solution and phase portrait.

4 Visual comparison of nonlinear and linearised phase portraits

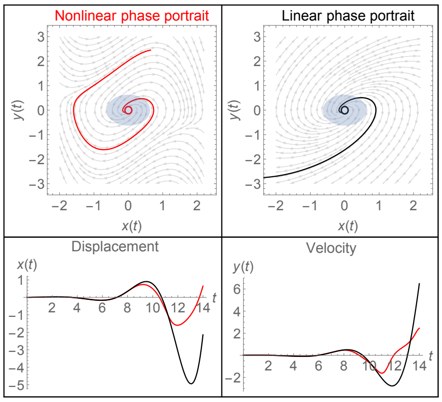
How much do the linearised and original nonlinear 2-D systems differ? We know that they are reasonably similar in close proximity to their respective fixed points, but what happens farther away? Let's consider the following examples. The following numerical file compares a nonlinear phase portrait of the Liénard equation to its linearised counterpart.

NUMERICS: NB#4

Interactive comparison of dynamics of the Liénard equation and its linearisation.

SLIDES: 19, 20

Nonlinear vs. linearised phase portrait



Example: The Liénard equation (red) and its linearisation (black). Parameter $\mu = 0.95$.

See Mathematica .nb file uploaded to the course webpage.

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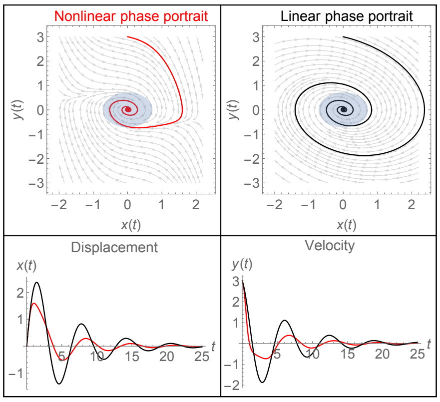
Nonlinear vs. linearised phase portrait



Example: The Liénard equation (red) and its linearisation (black). Parameter $\mu = -0.33$.

See Mathematica .nb file uploaded to the course webpage.

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Linearised systems are approximately equal to the full nonlinear systems only in close proximity to their respective fixed points. The regions of close proximity are shown with the spatially exaggerated blue circular areas.

5 Conservative systems

Below we prove that energy is conserved in a conserved mechanical system. Also, we provide a general definition of **conserved quantity** present in the conservative dynamical systems.

SLIDES: 21–23

Conservative system

Consider a system with one degree of freedom given by an equation of motion in the form

$$m\ddot{x} = F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}, \tag{24}$$

where m is the mass, V is the potential, and where force F is explicitly independent of times t (driving force) and $\dot{x}$ (attenuation/ damping).

In the conservative system the total energy is constant in time:

$$E = \frac{m\dot{x}^2}{2} + V(x) = \text{const}. \tag{25}$$

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Conservative system, conserved quantity

Definition: Given a system $\dot{\vec{x}} = \vec{f}(\vec{x})$, a **conserved quantity** is a real-valued continuous function $E(\vec{x})$ that is constant on the system trajectories, i.e., $\mathrm{d}E/\mathrm{d}t = 0$.
To avoid trivial examples, we also require that $E(\vec{x})$ be non-constant on every open set. Otherwise a constant function like $E(\vec{x}) = 0$ would qualify as a conserved quantity for every system, and so every system would be conservative! Our caveat rules out this silliness.

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Conservative system

Proof from a classical mechanics textbook: Using Eq. (24) we write

$$m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x} = 0 \quad \Big| \cdot \dot{x}, \tag{26}$$

$$m\ddot{x}\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} = 0. \tag{27}$$

The left-hand side of (27) is a so called perfect derivative (an exact time-derivative). By applying the chain rule ($\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$) in reverse we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m\dot{x}^2}{2} + V(x)\right) = 0, \tag{28}$$

from here it is clear that the sum of kinetic and potential energy do not change in time. Energy E is indeed a **conserved quantity**

$$\dot{E} = 0. \tag{29}$$

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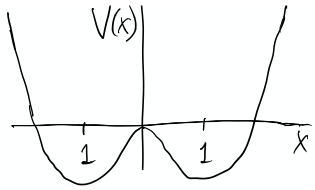


Figure 5: Potential function for the particle in the double-well potential problem.

5.1 Example: Particle in a double-well potential

5.1.1 Equation of motion and linear analysis

The potential is given by

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4. \tag{25}$$

Figure 5 shown the shape of the potential function. Using the Newton's second law we write

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{2}x^2 + \frac{1}{4}x^4\right), \tag{26}$$

where for simplicity we assume that mass $m = 1$. Notice, we are assuming that $F = -\mathrm{d}V/\mathrm{d}x$ is independent of both $\dot{x}$ and t; hence there is no damping or friction of any kind, and no time-dependent driving forces. The governing equation of motion simplifies to

$$\ddot{x} = x - x^3. \tag{27}$$

The governing equation as a system of first-order ordinary differential equations (ODE) for particle velocity $y = \dot{x}$ is in the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3. \end{cases} \tag{28}$$

Next, we analyse the system using linearisation. Fixed points of the system are found by solving

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} y^* = 0, \\ x^* - x^{*3} = 0, \end{cases} \tag{29}$$

for x^* and y^*. The fixed points are the following: $(x^*, y^*) = (\pm 1, 0)$, $(0, 0)$. The Jacobian matrix of Sys. (28) has the form

$$J = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}. \tag{30}$$

The Jacobian matrices evaluated at fixed points (x^*, y^*) are the following:

$$J\big|_{(\pm 1, 0)} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \tag{31}$$

here $\Delta = (0 \cdot 0) - (1 \cdot -2) = 0 + 2 = 2$ and $\tau = 0 + 0 = 0$. Based on the classification graph these fixed points are **centres**. These centres will prove to be true nonlinear centres because the system is conservative;

$$J\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{32}$$

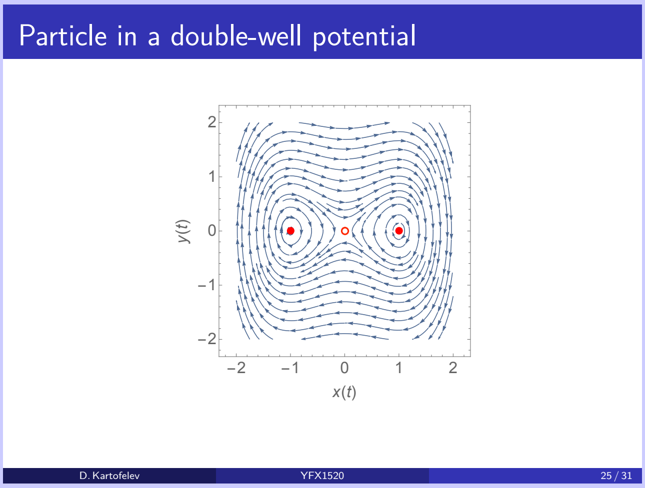
here $\Delta = (0 \cdot 0) - (1 \cdot 1) = 0 - 1 = -1$. Based on the classification graph this fixed point is a **saddle**. We construct a phase portrait by combining the above results into a single graph or by using a computer to do it for us.

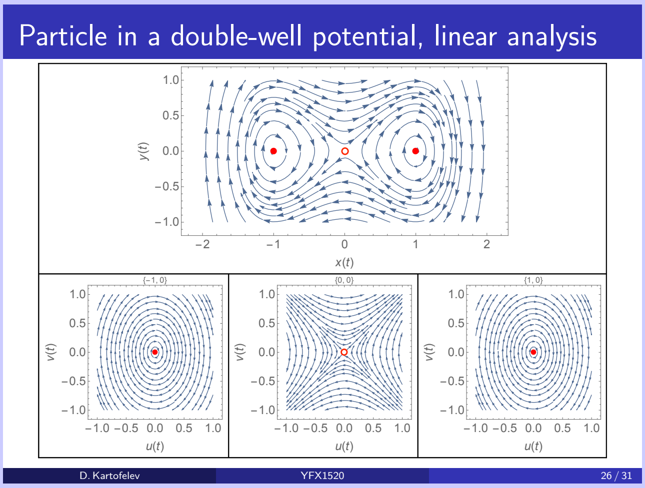
NUMERICS: NB#5

Particle in a double-well potential, conservative system and homoclinic orbits. Linear analysis of fixed points, numerical solution and phase portrait.

Numerical calculation of results (29)–(32) are also presented here.

SLIDES: 25–27





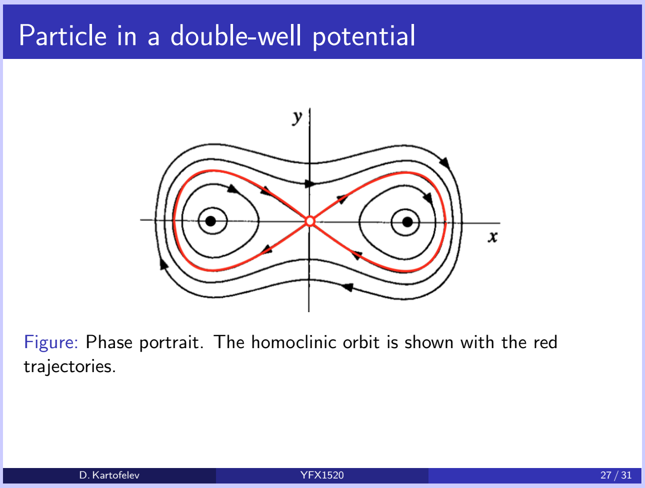


Figure: Phase portrait. The homoclinic orbit is shown with the red trajectories.

We have encountered a new type of trajectory called the **homoclinic orbit**. Perturbed trajectory starting at the saddle point returns to the same fixed point where the process may/can start over again when perturbed.

NUMERICS: NB#5

Particle in a double-well potential, conservative system and homoclinic orbits. Linear analysis of fixed points, numerical solution and phase portrait.

The phase portraits shown on the above lecture slides.

5.1.2 The Hamiltonian and the system phase trajectories

In conservative systems the phase portrait trajectories are closed curves defined by *contours* of constant energy

$$E = \underbrace{\frac{m\dot{x}^2}{2}}_{\text{kin. en.}} \underbrace{-\frac{1}{2}x^2 + \frac{1}{4}x^4}_{\text{pot. en.}} = \text{const.} \tag{33}$$

For simplicity we select $m = 1$ and for velocity $y = \dot{x}$ we write

$$E(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4, \tag{34}$$

where x-axis corresponds to the potential energy and y-axis to the kinetic energy. This surface defined by the Hamiltonian (33) is shown in the following numerical file and on Slide 28.

NUMERICS: NB#5

Particle in a double-well potential, conservative system and homoclinic orbits. Linear analysis of fixed points, numerical solution and phase portrait.

SLIDE: 28

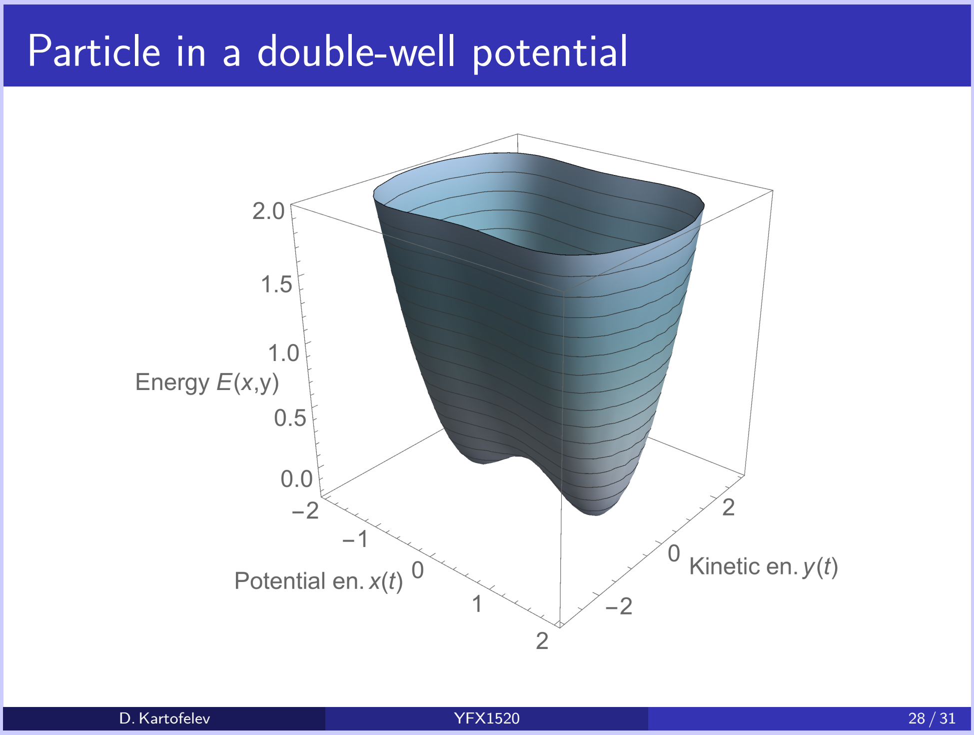


Figure 6 shows the connection between conserved energy E, the Hamiltonian and selected phase portrait trajectories by plotting four phase portrait trajectories corresponding to three different values of constant energy E.

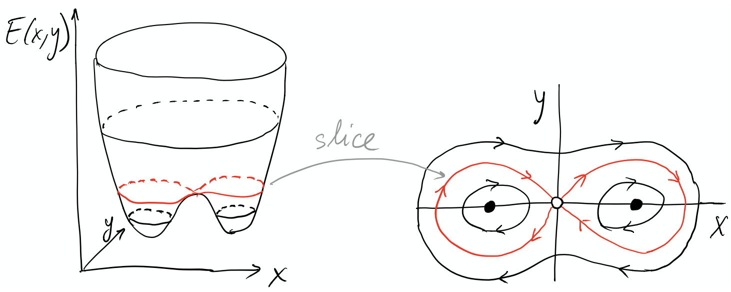


Figure 6: Four trajectories corresponding to different values of energy E. The homoclinic orbit and the corresponding energy E level are shown with the red curves/trajectories.

Revision questions

1. Provide an example of nonlinear 2-D system.
2. Explain linearisation of 2-D systems about fixed points.
3. Can all nonlinear systems be linearised with the aim of identifying their fixed point type?
4. Linearise the following system

$$\begin{cases} \dot{x} = 4x - 4xy, \\ \dot{y} = -9y + 18xy. \end{cases} \tag{35}$$

5. Without taking derivatives, linearise the following systems:

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x, \end{cases} \tag{36}$$

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + y^2. \end{cases} \tag{37}$$

6. Define Jacobian matrix of a system.
7. Sketch a homoclinic orbit.
8. Define conservative dynamical system.