Lecture 3: Over-damped bead on a rotating hoop, dimensional analysis and dimensionless form of equations of motion, introduction to 2-D systems, uniqueness of solution and phase space trajectories

Contents

Demonstration: (Over-damped) bead on a rotating hoop.

1 Introduction

We continue our discussion on bifurcation analysis of dynamical systems. A majority of time of this lecture will be spent on an analysis of a relatively intuitive mechanics problem—the over-damped bead on a rotating rigid hoop. This problem provides an example of a bifurcation in a Newtonian mechanical system. It also illustrates the subtleties involved in replacing Newton's law, which is a **second-order equation**, by a simpler **first-order equation**.

Figure 1: The hoop and the sliding bead used during the classroom demonstration.

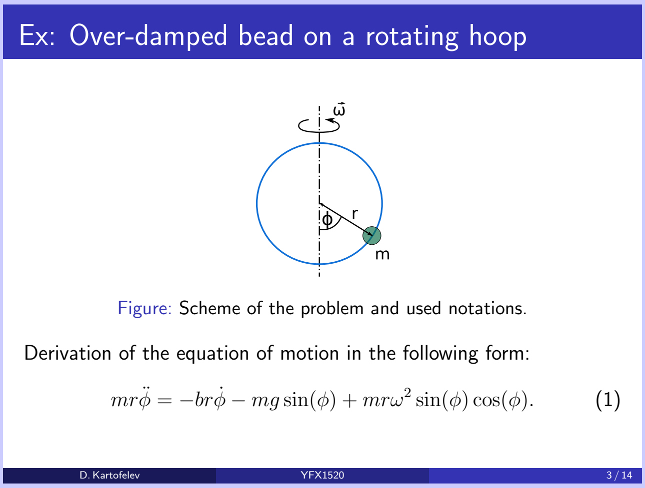
The bifurcation analysis of the aforementioned problem will lead us to the notion of 2-D phase portraits—an important topic[1] we will continue introducing/exploring further in the next lecture. Figure 1 shows the bead on the hoop used during the lecture.

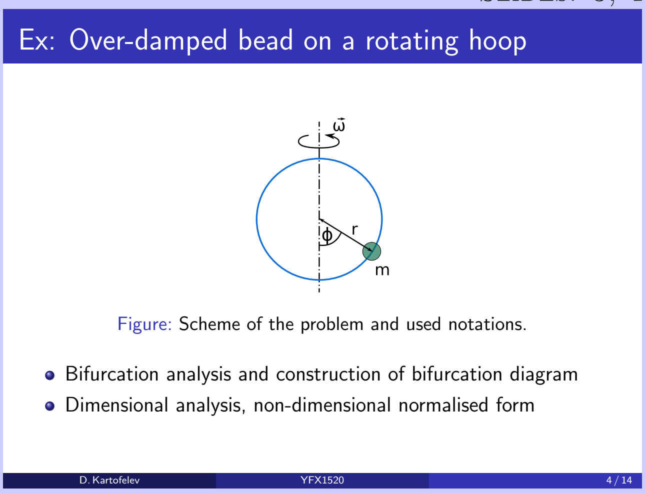
2 Over-damped bead on a rotating hoop

2.1 Equation of motion and simplification of the problem

Let's derive the equation of motion for the **over-damped bead on a rotating hoop** problem. We start with the simplified dynamics of the **bead on a rotating hoop** and explain the over-damping phenomena as we go along. Let's use the simplest and most familiar approach known to us for doing so—we determine all the forces acting on the bead and apply the Newton's second law of motion.

SLIDES: 3, 4





The above slides shows the schematic drawing of the bead on the hoop problem. The bead of mass m slides along a wire hoop of radius r. The hoop is constrained to rotate at a constant angular velocity ω about its vertical axis. Our goal is to analyse the motion of the bead, given that it is acted on by both **gravitational** and ***centrifugal*** forces. This is the classical statement of the problem, but now we want to add a new twist: suppose that there's also a **frictional force** on the bead that opposes its motion. To be specific, imagine that the whole system is immersed in a vat of molasses or some other

[1] Relevant to Your coursework.

very viscous fluid, and that the friction is due to **viscous damping**. We let ϕ be the angle between the bead and the downward vertical direction. By convention, we restrict ϕ to the range $-\pi < \phi \leq \pi$, so there is only one angle for each point on the hoop. Although, on some figures below we will show the rage to be $-2\pi < \phi \leq 2\pi$ to demonstrate the periodicity of the obtained solutions.

Before we continue, we also let

$$\rho = r \sin \phi, \tag{1}$$

denote the distance of the bead from the vertical axis. Then the coordinate framework of the problem is as shown in Fig. 2. Since, the hoop is assumed to be **rigid**, we only have to resolve the forces within the plane of the hoop and along the **tangential direction** at point m. Figure 2 shows how we do this. Essentially, we are deriving the equation of motion of mass m in the **co-rotating** with the hoop **framework**, i.e. the plane defined by the hoop.

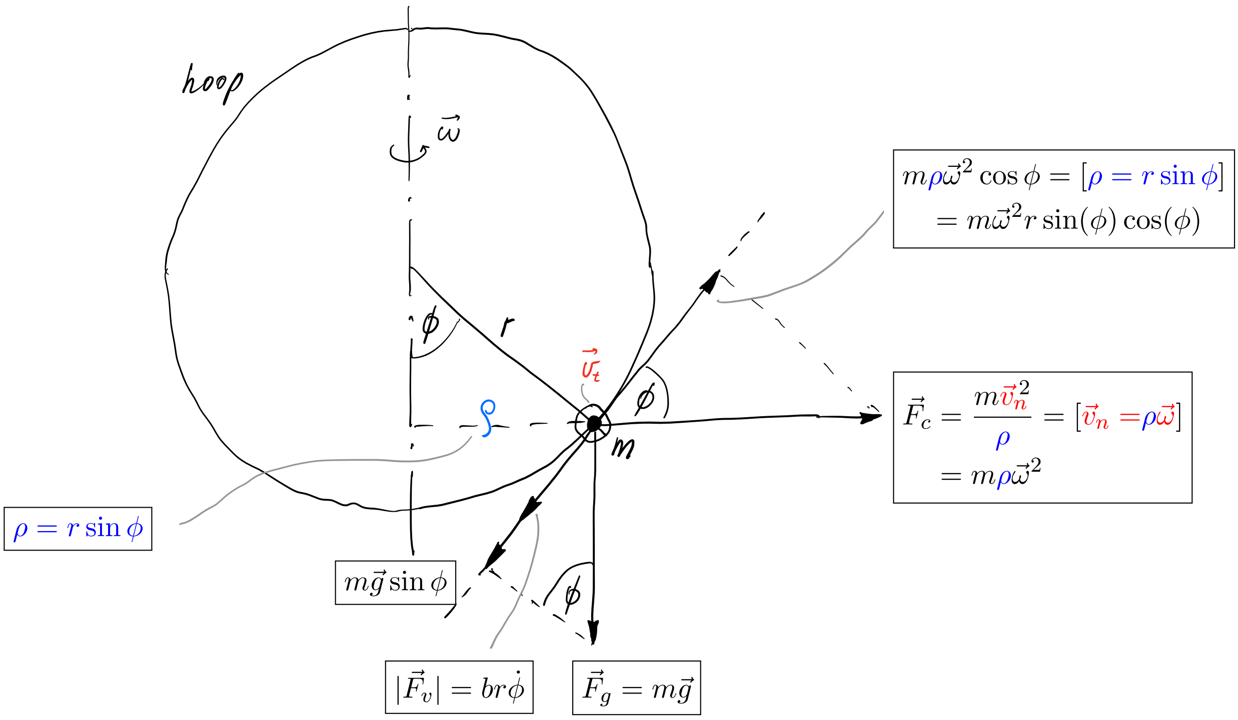


Figure 2: Forces acting on the bead with mass m. All forces are projected onto a tangential direction at the point m. The problem is solved in the co-rotating with the hoop framework.

Now we write Newton's law for the bead. There is a downward **gravitational** force $\vec{F}_g = m\vec{g}$, a sideways ***centrifugal*** force $\vec{F}_c = (m\vec{v}_n^2)/\rho = m\rho\vec{\omega}^2$, and a **tangential damping** force $|\vec{F}_v| = br\dot{\phi}$, where b is the viscous damping or friction constant. Constants g and b are taken to be positive; negative signs will be added later as needed. After substituting $\vec{v}_n = \rho\vec{\omega}$ and $\rho = r\sin\phi$ into the *centrifugal* term, and recalling that the **tangential acceleration** $|\vec{a}| = r\ddot{\phi}$, we obtain the governing equation

$$m\vec{a} = \sum_i \vec{F}_i, \tag{2}$$

where $\vec{F}_i$ are the forces acting in the tangential direction and the force projections onto the tangent line. More explicitly

$$\underbrace{mr\ddot{\phi}}_{\substack{\text{inertial}\\ \text{term}}} = \underbrace{-br\dot{\phi}}_{\substack{\text{viscous}\\ \text{or}\\ \text{friction}\\ \text{term}}} - \underbrace{m|\vec{g}|\sin(\phi)}_{\substack{\text{gravity}\\ \text{term}}} + \underbrace{mr|\vec{\omega}|^2\sin(\phi)\cos(\phi)}_{\substack{\text{centrifugal}\\ \text{term}}}. \tag{3}$$

The viscous and gravity terms work against the centrifugal term, hence the minus sign in front of them. After evaluating the vector magnitudes and after some algebraic simplifications we get

$$mr\ddot{\phi} = -br\dot{\phi} - mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi), \tag{4}$$

or equivalently

$$\boxed{mr\ddot{\phi} = -br\dot{\phi} + mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right).} \tag{5}$$

This is a **second-order differential equation**, as all Newton's laws are, since the second derivative $\ddot{\phi}$ is the highest one that appears. We are not yet equipped to analyse second-order equations, so we would like to find some conditions under which we can safely neglect inertial term $mr\ddot{\phi}$. This would allow for Eq. (5) to be reduced to a first-order equation that we know how to analyse. Of course, this is a dicey business: we can't just neglect terms because we feel like it! But we will for now, and later at the end of this lecture we'll try to find a regime where our approximation is justified/valid. After neglecting inertial term $mr\ddot{\phi}$ we get

$$\boxed{br\dot{\phi} = mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right).} \tag{6}$$

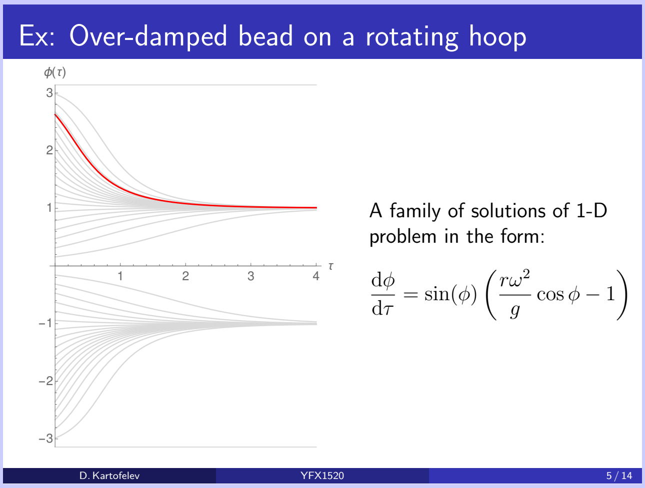
Now that we have the equation we can study it graphically using a computer. The following interactive numerical file includes the 1-D phase portrait defined by Eq. (6).

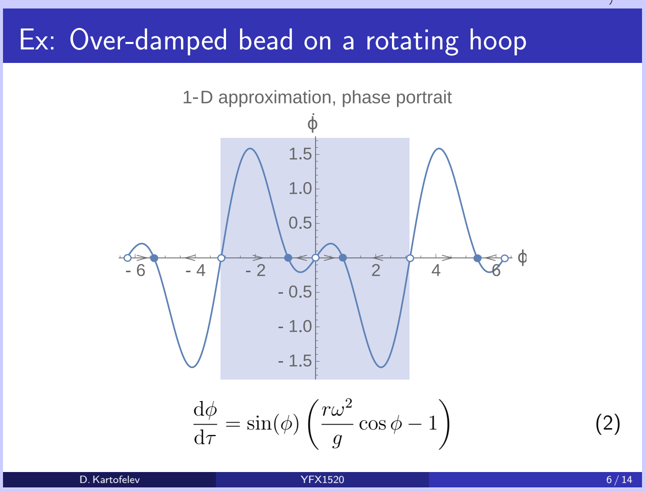
Numerics: nb#1

1-D approximation corresponding to the over-damped bead on a hoop dynamics: the 1-D phase portrait and a family of numerical solutions corresponding to a set of initial conditions.

This file considers dimensionless and normalised form of Eq. (6). The derivation of the dimensionless form of Eq. (6) namely Eq. (13) is explained in the blue text box for Slides 5 and 6 (see below).

Slides: 5, 6





As was discussed in previous lectures **phase portrait** is a powerful tool for studying dynamical systems without directly integrating them. It is often beneficial to study equations in a dimensionless and normalised form.

——— *Skip if needed:* start ———

Independent variable t of Eq. (6) has a time dimension (seconds, in SI system of units). It is made dimensionless by introducing dimensionless time τ and using variable exchange in the form

$$\tau = \frac{t}{T}\ [1] \quad \Rightarrow \quad t = \tau T\ [\mathrm{s}], \tag{7}$$

where T is the time scale related to the flow on the line defined by Eq. (6). We substitute variable exchange (7) into Eq. (6) to get

$$br\frac{d\phi}{d(\tau T)} = mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right), \tag{8}$$

$$\frac{br}{T}\cdot\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \tag{9}$$

The obtained balance of forces is made dimensionless by dividing it with a typical force mg [N] present in the equation

$$\frac{br}{T}\cdot\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right) \quad \Big| \div mg, \tag{10}$$

$$\frac{br}{mgT}\cdot\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \tag{11}$$

Additionally, we select the characteristic time scale T as follows (the selection criteria is explained in greater detail in Sec. 2.3)

$$\frac{br}{mgT} = O(1) \approx 1 \quad \Rightarrow \quad T = \frac{br}{mg}, \tag{12}$$

to get the desired dimensionless and normalised form of the original equation

$$\boxed{\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right).} \tag{13}$$

——— *Skip if needed:* stop ———

2.2 Bifurcation analysis of the simplified 1-D problem

From now on our concern is with the first-order system in the form (6). The fixed points of (6) correspond to equilibrium positions for the bead. Naturally, we are interested in the stability of the fixed points. We find the fixed points by solving

$$\dot{\phi} = 0 \quad \Rightarrow \quad \frac{mg}{br}\sin(\phi^*)\left(\frac{r\omega^2}{g}\cos\phi^* - 1\right) = 0. \tag{14}$$

Firstly, there are alway two fixed point corresponding to

$$\frac{mg}{br}\sin\phi^* = 0 \quad \Rightarrow \quad \sin\phi^* = 0 \quad \Rightarrow \quad \begin{cases}\phi^* = 0,\\ \phi^* = \pm\pi.\end{cases} \tag{15}$$

The bottom of the hoop ($\phi^* = 0$) and the top ($\phi^* = \pm\pi$). Secondly, and more interestingly, there are two additional fixed points that depends on angular velocity $\omega > 0$ and are given by

$$\frac{r\omega^2}{g}\cos\phi^* - 1 = 0. \tag{16}$$

Below it will prove beneficial to rename quantity $r\omega^2/g$ as follows

$$\gamma = \frac{r\omega^2}{g}, \tag{17}$$

and use γ, the parameter that is strongly proportional to angular velocity ω of the hoop, to rewrite (16)

$$\gamma\cos\phi^* = 1, \tag{18}$$

$$\cos\phi^* = \frac{1}{\gamma}. \tag{19}$$

The obtained result can be studied graphically using an arbitrary y-axis as was described in Lecture 2. We plot $\cos\phi$ vs. ϕ graph ($y = \cos\phi$) and look for its intersections with the constant function $1/\gamma$ ($y = 1/\gamma$), shown by the horizontal dashed lines in Fig. 3. For $\gamma < 1$ there are no intersections, whereas for $\gamma > 1$ there

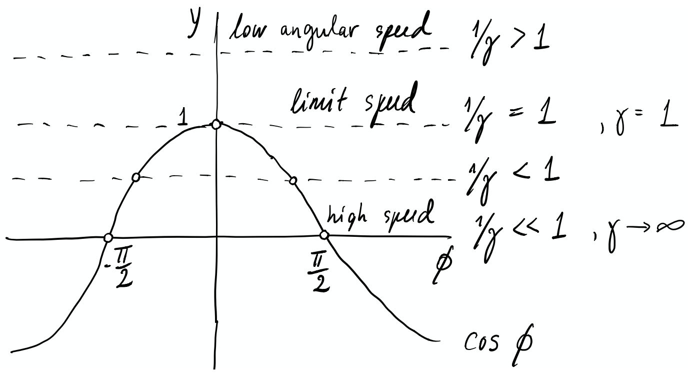


Figure 3: Graphical study of Eq. (19). Intersection of $y = \cos\phi$ and $y = 1/\gamma$. For $\gamma > 1$ two intersection points appear, shown with the hollow bullets.

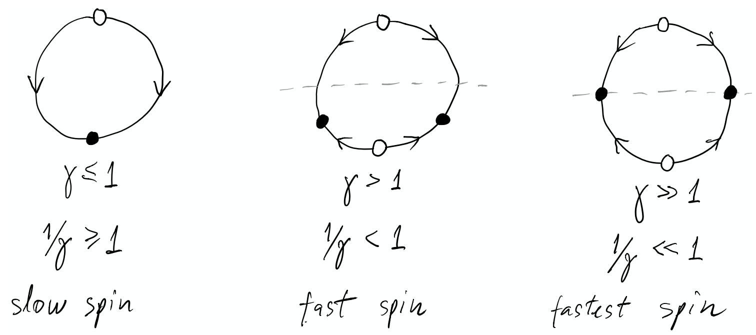


Figure 4: Fixed points as they appear on the hoop. Stable fixed points are shown with the filled bullets and unstable ones with the hollow bullets. (Left) Case where $\gamma \leq 1$. (Middle) Case where $\gamma > 1$. (Right) Case where $\gamma \gg 1$.

is a symmetrical pair of intersections to either side of $\phi^* = 0$. As $\gamma \to \infty$, these intersections approach $\phi^* = \pm\pi/2$. Figure 4 plots fixed points ϕ^* as they appear on the hoop for the cases $\gamma \leq 1$ and $\gamma > 1$.

To summarise our results so far, we plot the **bifurcation diagram** by plotting all the fixed points as a function of parameter γ. Figure 5 shows the resulting diagram. As usual, the solid lines and curves denote the stable fixed points and dashed lines the unstable ones. We see that a **supercritical pitchfork bifurcation** occurs when $\gamma = 1$ (bifurcation point). It's left to you to check the stability of the fixed points, using linear stability analysis or the graphical approach, i.e. the slope values evaluated at fixed points ϕ^*, see Slide 6.

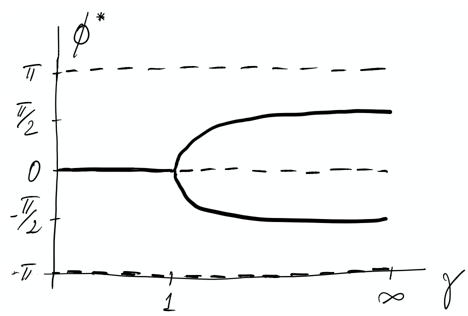


Figure 5: Bifurcation diagram for the bead on the rotating hoop problem. Bifurcation parameter γ is proportional to angular velocity ω. Stable fixed points are indicated by the solid lines and curves, and unstable fixed points by the dashed lines.

The following numerical file contains an interactive code that plots the intersection events shown in Fig. 3 and a quantitatively accurate rendering of the bifurcation diagram shown in Fig. 5.

NUMERICS: NB#2

Over-damped bead on a hoop dynamics: graphical determination of the fixed points and the bifurcation diagram (first-order model).

The quantitatively accurate rendering of the bifurcation diagram shown in Fig. 5.

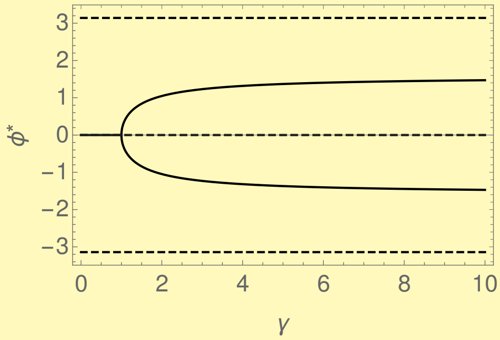


Figure 6: Quantitatively accurate bifurcation diagram for the bead on the rotating hoop problem. Bifurcation parameter γ is strongly proportional to angular velocity ω. The stable fixed points are indicated by the solid lines and curves, and the unstable fixed points by the dashed lines.

Here's the physical interpretation of the results: When $\gamma < 1$, the hoop is rotating slowly and the centrifugal force is too weak to balance the force of gravity. Thus the bead slides down to the bottom $\phi = 0$ and stays there. But if $\gamma > 1$, the hoop is spinning fast enough that the bottom becomes unstable. Since the centrifugal force grows as the bead moves farther from the bottom, any slight displacement of the bead will be amplified. The bead is therefore pushed up the hoop until gravity balances the centrifugal force; according to (19), and Figs. 3, 5 and 6 this balance occurs at

$$\phi^* = \pm \lim_{\gamma\to\infty} \left[\cos^{-1}\left(\frac{1}{\gamma}\right)\right] = \pm\frac{\pi}{2}. \tag{20}$$

Which of these two fixed points is actually selected depends on the initial disturbance. Even though the two fixed points are entirely symmetrical, a slight asymmetry in the initial conditions will lead to one of them being chosen—physicists sometimes refer to these as **symmetry-broken** solutions. In other words, the actualised solution has less symmetry than the governing equation.

What is the symmetry of the governing equation? Clearly the left and right halves of the hoop are physically equivalent—this is reflected by invariance of Eqs. (3), (5) and (6) under the change of variables $\phi \to -\phi$. As we mentioned in Lecture 2, the pitchfork bifurcations are to be expected in situations where such a symmetry exists.

2.3 Dimensional analysis of the full 2-D model, dimensionless form

We use Eq. (5) to determine under which conditions, i.e. parameter values, our prior assumption that inertial term $mr\ddot{\phi}$ can be neglected holds true. The dimensional analysis and the dimensionless form of the equation of motion allows us to compare different terms of the equation to each-other. The advantage of the dimensionless formulation is that we know how to define *small* terms, i.e., "much less than one." Furthermore, non-dimensionalising the equation reduces the number of parameters by lumping them together into **dimensionless groups**. This reduction always simplifies the analysis. We rewrite Eq. (5) using Leibniz notation to be more transparent and accurate with our representation of the derivatives present. The governing equation takes the following form:

$$\underbrace{mr\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}}_{[\mathrm{kg}][\mathrm{m}]\frac{[1]}{[\mathrm{s}^2]}\equiv[\mathrm{N}]} = \underbrace{-br\frac{\mathrm{d}\phi}{\mathrm{d}t}}_{\frac{[\mathrm{kg}]}{[\mathrm{s}]}[\mathrm{m}]\frac{[1]}{[\mathrm{s}]}\equiv[\mathrm{N}]} + \underbrace{mg\sin(\phi)}_{[\mathrm{kg}]\frac{[\mathrm{m}]}{[\mathrm{s}^2]}\equiv[\mathrm{N}]}\underbrace{\left(\frac{r\omega^2}{g}\cos\phi - 1\right)}_{\frac{[\mathrm{m}][1][\mathrm{s}^2]}{[\mathrm{s}^2][\mathrm{m}]}\equiv[1]}, \tag{21}$$

here and below the square brackets are used to show the dimensions associated with the different parts of the equation. In order to non-dimensionalise or simply eliminate the dimension of the only independent variable that is time t, we introduce dimensionless time τ that has the following form:

$$\tau = \frac{t}{T}\ [1] \quad \Rightarrow \quad t = \tau T\ [\mathrm{s}], \tag{22}$$

where T is an arbitrary time period (a characteristic time scale to be chosen later). When T is chosen correctly, the new derivatives $\mathrm{d}\phi/\mathrm{d}\tau$ and $\mathrm{d}^2\phi/\mathrm{d}\tau^2$ should be $O(1)$, i.e., of order unity. After substituting (22) into (21) we get

$$mr\frac{\mathrm{d}^2\phi}{\mathrm{d}(\tau T)^2} = -br\frac{\mathrm{d}\phi}{\mathrm{d}(\tau T)} + mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \tag{23}$$

Constants T and T^2 can be factored out

$$\frac{mr}{T^2}\cdot\underbrace{\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}}_{\frac{[1]}{[1]^2}\equiv[1]} = -\frac{br}{T}\cdot\underbrace{\frac{\mathrm{d}\phi}{\mathrm{d}\tau}}_{\frac{[1]}{[1]}\equiv[1]} + mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \tag{24}$$

At this point the derivative terms are indeed dimensionless. In order to eliminate the rest of the dimensions we divide by characteristic force mg present in the right-hand side of the equation

$$\underbrace{\frac{mr}{T^2}}_{[\mathrm{N}]}\cdot\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -\underbrace{\frac{br}{T}}_{[\mathrm{N}]}\cdot\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \underbrace{mg\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right)}_{[\mathrm{N}]} \quad \Big| \div \underbrace{mg}_{[\mathrm{N}]}, \tag{25}$$

to get the desired dimensionless form of the entire equation

$$\underbrace{\frac{\cancel{m}r}{\cancel{m}gT^2}}_{[1]}\cdot\underbrace{\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}}_{[1]} = -\underbrace{\frac{br}{mgT}}_{[1]}\cdot\underbrace{\frac{\mathrm{d}\phi}{\mathrm{d}\tau}}_{[1]} + \underbrace{\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right)}_{[1]}. \tag{26}$$

The magnitudes of all the terms of this equation can now be compared to each other. In order for the assumption regarding the inertial term, presented in Sec. 2.1, to hold we select time scale T such that

$$\underbrace{\frac{r}{gT^2}}_{\ll O(1)}\cdot\underbrace{\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}}_{O(1)} = -\underbrace{\frac{br}{mgT}}_{O(1)}\cdot\underbrace{\frac{\mathrm{d}\phi}{\mathrm{d}\tau}}_{O(1)} + \underbrace{\sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right)}_{O(1)}. \tag{27}$$

We are interested in the regime where the left-hand side of Eq. (27) is negligible compared to all the other terms, and where all the terms on the right-hand side are of comparable size, ideally $O(1)$. Additionally, we are *allowed* to assume that derivatives $\mathrm{d}\phi^2/\mathrm{d}\tau^2$ and $\mathrm{d}\phi/\mathrm{d}\tau$ are also $O(1)$. This implies that time scale T and initial conditions $\phi(0)$ and $\mathrm{d}\phi(0)/\mathrm{d}\tau$ are chosen *appropriately*. Using the first term on the right-hand side of Eq. (27) we resolve T as follows

$$\frac{br}{mgT} = O(1) \approx 1 \quad \Rightarrow \quad T = \frac{br}{mg}\ [\mathrm{s}]. \tag{28}$$

This selection of T results in the following equation

$$\frac{\cancel{r}m^2g^{\cancel{2}}}{\cancel{g}b^2r^{\cancel{2}}}\cdot\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -\frac{\cancel{brmg}}{\cancel{mgbr}}\cdot\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \sin(\phi)\left(\frac{r\omega^2}{g}\cos\phi - 1\right), \tag{29}$$

$$\underbrace{\frac{m^2g}{b^2r}}_{[1]}\cdot\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \sin(\phi)\left(\underbrace{\frac{r\omega^2}{g}}_{[1]}\cos\phi - 1\right). \tag{30}$$

Renaming the new-found dimensionless group located on the left-hand side of the result to

$$\alpha = \frac{m^2g}{b^2r}, \tag{31}$$

and remembering that dimensionless group $r\omega^2/g$ (17) is called γ results in

$$\boxed{\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \sin(\phi)\left(\gamma\cos\phi - 1\right).} \tag{32}$$

This equation has only two dimensionless groups α and γ instead of the five parameters (m, r, b, g, ω) present in Eq. (5)—our starting point.

Let's analyse the obtained result further and try to understand under which physical condition, i.e. under which values of parameter, it is reasonable to neglect inertial term $mr\ddot{\phi}$ present in the initial equation (5). According to the working assumptions

$$\alpha \ll O(1) \approx 1 \quad \Rightarrow \quad \frac{m^2 g}{b^2 r} \ll O(1) \approx 1, \tag{33}$$

which means that

$$m^2 \ll \frac{b^2 r}{g} \quad \text{or} \quad b^2 \gg \frac{m^2 g}{r}. \tag{34}$$

Since powers, m^2 and b^2, are more dominant/prominent algebraically speaking, we focus on them. For high viscosity $b \gg 1$ and for small mass $m \ll 1$ the initial assumption holds true. Now we see what over-damping means. Mass m has to be strongly damped by viscosity b or mass m has to be very small. In summary, the dimensional analysis suggests that in the over-damped limit $\alpha \to 0$ Eqs. (5) or (32) can be well approximated by the first-order system given by Eqs. (6) or (13).

Unfortunately, there is something fundamentally wrong with our idea of replacing a second-order equation by a first-order equation. The trouble is that a second-order equation requires two initial conditions, whereas a first-order equation has only one. In our case, the bead's motion is determined by its **initial angular position** ϕ and **angular velocity** $\mathrm{d}\phi/\mathrm{d}\tau$, in the dimensionless case. These two quantities can be chosen completely **independent of each other**. But that's not true for the first-order system: given the initial position, the initial velocity is dictated by Eqs. (6) or (13). Thus the solution to the first-order system will not, in general, be able to satisfy both initial conditions. Imagine that someone gives our bead a very high absolute initial velocity, so that, e.g., in the dimensionless case

$$\frac{\mathrm{d}\phi(0)}{\mathrm{d}\tau} \neq \sin(\phi(0))(\gamma\cos\phi(0) - 1). \tag{35}$$

We seem to have run into a paradox. Is our simplified 1-D model valid in the over-damped limit or not? If it is valid, how can we satisfy the two arbitrary and mutually independent initial conditions demanded by the full 2-D model given by Eqs. (5) or (32)? Once again the graphical approach comes to the rescue. An answer to this conundrum can be given by graphing and analysing the 2-D phase portrait.

3 Introduction to 2-D phase portrait

In previous lectures we have exploited the idea that first-order system $\dot{x} = f(x)$ can be regarded as a vector field on a line (flow on a line). By analogy, a second-order system can be regarded as a **vector field on a plane**—a **phase plane**. By introducing variable exchange $\hat{\psi} = \dot{\phi}$ (angular velocity, do not confuse with ω) we rewrite the second-order model, given by Eq. (5), as a system of first-order ODEs in the following form:

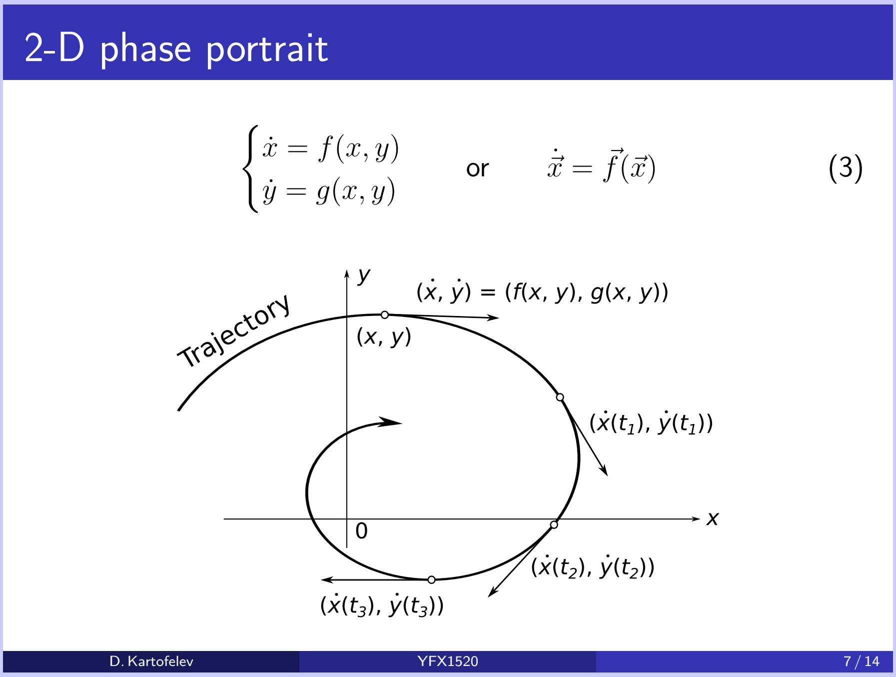
$$\begin{cases} \dot{\phi} = \hat{\psi} \\ mr\dot{\hat{\psi}} = -br\hat{\psi} + mg\sin(\phi)\left(\dfrac{r\omega^2}{g}\cos\phi - 1\right) \end{cases} \Rightarrow \begin{cases} \dot{\phi} = \hat{\psi}, \\ \dot{\hat{\psi}} = -\dfrac{b}{m}\hat{\psi} + \dfrac{g}{r}\sin(\phi)\left(\dfrac{r\omega^2}{g}\cos\phi - 1\right). \end{cases} \tag{36}$$

Same can be done for the dimensionless and normalised form (32) derived in the previous section. We introduce variable exchange $\psi = \mathrm{d}\phi/\mathrm{d}\tau$ and rewrite the equation in the following form:

$$\begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = \psi \\ \alpha\dfrac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\psi + \sin(\phi)\left(\gamma\cos\phi - 1\right) \end{cases} \Rightarrow \begin{cases} \dfrac{\mathrm{d}\phi}{\mathrm{d}\tau} = \psi, \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\dfrac{1}{\alpha}\psi + \dfrac{1}{\alpha}\sin(\phi)\left(\gamma\cos\phi - 1\right). \end{cases} \tag{37}$$

These systems can be numerically integrated using standard ODE solvers. A way to think about these systems graphically in the context of 2-D phase plane/portrait is explained below.

SLIDE: 7



A phase portrait of a homogeneous second-order system represented here in the general form:

$$\begin{cases} \dot{x} = f(x, y), \\ \dot{y} = g(x, y), \end{cases} \tag{38}$$

is constructed by plotting vector field $\dot{\vec{x}} = (\dot{x}, \dot{y})^T = (f(x, y), g(x, y))^T$ against the independent variables x and y in xy-plane, as shown in Fig. 7 and on Slide 7. A **trajectory** of the vector field represents a single solution of Sys. (38) corresponding to an initial condition—a single point in the plane (x_0, y_0), where $x_0 = x(0)$, $y_0 = y(0)$.

It is helpful to visualise the vector field in terms of the motion of an imaginary fluid. We imagine that a fluid is flowing steadily on the phase plane with a local velocity given by Sys. (38). Then, to find the trajectory starting at initial condition (x_0, y_0), we place an imaginary particle or phase point at (x_0, y_0) and *watch* how it is carried around by the flow.

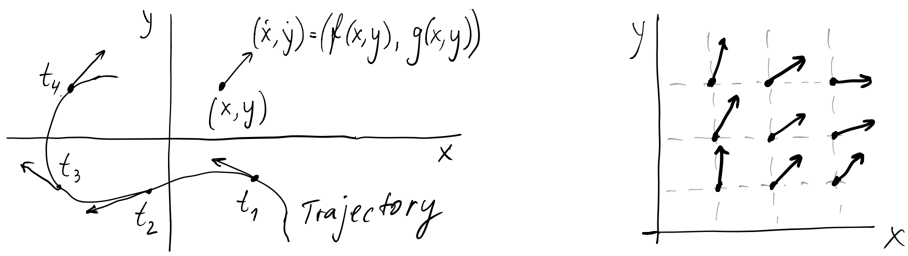


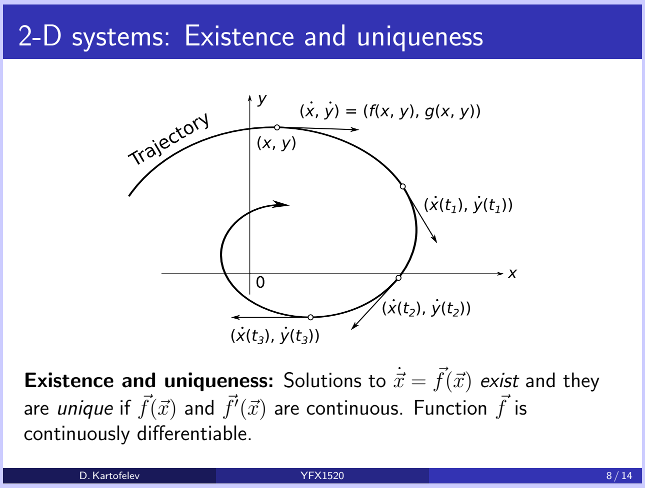
Figure 7: (Left) Phase portrait of a 2-D or a second-order problem. A trajectory is shown with the continuous curve where $t_1 < t_2 < t_3 < t_4$. (Right) An idea behind construction of a 2-D phase portrait—an entire vector field for the shown ranges of x and y. The vector field vectors are evaluated and shown for points (x, y) placed in a uniform grid that is shown with the dashed lines.

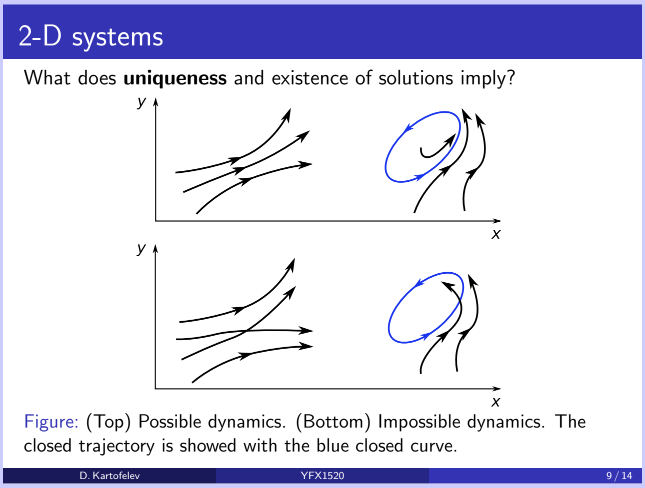
In this course we will be using computers to plot two and higher-dimensional phase portraits. Figure 7 (Right) shows how computer may constructs a phase portrait using simplest algorithms. The further explanation and introduction to the topic of 2-D phase portraits will resume in the next week's lecture.

3.1 Topological properties of 2-D phase portrait trajectories

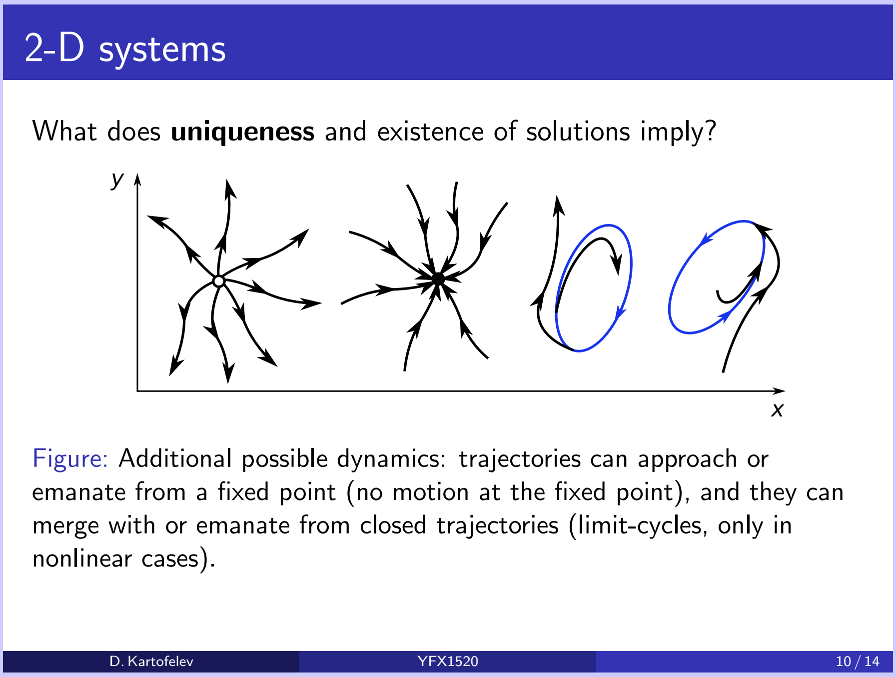
Slides: 8–10

Below, topological properties of 2-D phase portrait trajectories are presented.





The **uniqueness** implies that non-closed trajectories <u>can not have two futures</u> when starting at a selected initial condition—point (x_0, y_0), both in forward and backward time.



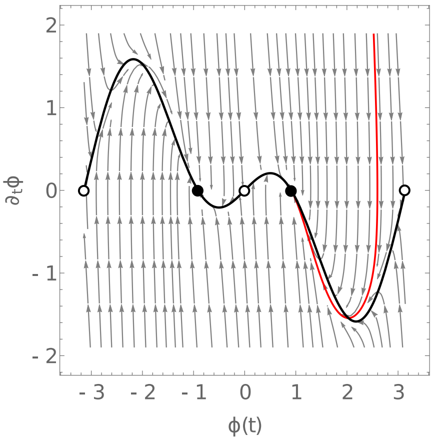
Trajectories can approach to or emanate from a single fixed point because they can not reach or escape them <u>in a finite time</u>, thus not contradicting the uniqueness of solutions property. Also, if you are located exactly at fixed point (x^*, y^*) you are not moving. The limit-cycle dynamics will be explained in the future lectures.

3.2 A second look at the over-damped bead on the rotating hoop dynamics

The paradox raised in Sec. 2.3 is clarified when plotting and overlapping the phase portraits of second-order full model (32) and simplified first-order model (13).

SLIDE: 11

Ex: Over-damped bead on a rotating hoop



Phase portrait of 2-D problem:

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}\phi}{\mathrm{d}\tau}+\sin(\phi)(\gamma\cos\phi-1), \quad (4)$$

where

$$\alpha = \frac{m^2 g}{b^2 r}, \quad (5)$$

$$\gamma = \frac{r\omega^2}{g}. \quad (6)$$

1-D phase portrait of $\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \sin(\phi)\,(\gamma\cos\phi - 1)$ is shown with black.

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The solid black curve corresponds to the dimensionless 1-D system defined by Eq. (13). The 2-D phase portrait calculated using the dimensionless model defined by Eq. (32) or Sys. (37) is shown with the grey vector field. A single trajectory of the second-order model is shown with the red curve. The stability of fixed points ϕ^* corresponding to the 1-D model are indicated by the filled and empty bullets. The fixed points of the 1-D model also coincide with fixed points of the 2-D model (ϕ^*, ψ^*), where $\psi = \mathrm{d}\phi/\mathrm{d}\tau$ (*cf.* Sys. (37)).

The interactive visual analysis of the 2-D phase portrait can be performed using the following numerical file.

NUMERICS: NB#3

Numerical solution and phase portrait of the over-damped bead on a hoop problem (second-order model).

The numeric integration is performed on the dimensionless form of the full model given by Sys. (37) using an ODE integrator (Runge-Kutta Method, RK45).

The visual/graphical analysis of the above phase portrait and its trajectories allows us to conclude that a typical 2-D trajectory is made of two parts: a **rapid initial transient**, shown on Slide 11 with the red trajectory, during which the trajectory zaps onto the curve defined by Eq. (13), followed by a much slower drift along this curve.

Now we see how the paradox discussed in Sec. 2.3 is resolved: Second-order models (5) or (37) do behave like the first-order models (6) or (13), but only after a rapid initial transient. During this transient, it is not correct to neglect inertial term $\mathrm{d}^2\phi/\mathrm{d}\tau^2$. The problem with our earlier approach is that we used only a single time scale $T = br/(mg)$, this time scale is characteristic of the slow drift process, but not of the rapid initial transient.

Reading suggestion

For a biological example of bifurcation, we turn to a model for the sudden outbreak of an insect called the spruce budworm. This insect is a serious pest in eastern Canada, where it attacks the leaves of the balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years. The insect outbreak model analysis can be found in the main textbook of the course cited below or on the Internet as a video lecture[2] given by Steven Strogatz himself.

Link	**File name**	**Citation**
Read#1	reading.pdf	S.H. Strogatz, *Nonlinear Dynamics and Chaos With Applications to Physics, Biology, Chemistry, and Engineering*, Perseus Books Publishing, L.L.C., pp. 73–79, (1994).

Revision questions

1. What is symmetry-broken solution?
2. What is dimensional analysis of an equation of motion?
3. What is dimensionless form of an equation?
4. What is normalised form of an equation?
5. How many initial conditions does first-order ODE have?
6. How many initial conditions does second-order ODE have?
7. Explain the notion of different time scales of a dynamical system.
8. Derive the dimensionless form of the following equation of motion:

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + ku = 0, \tag{39}$$

   where u is the displacement, m is the mass and t is the time. Additionally, determine the dimensions of damping coefficient b and stiffness k.
9. Derive the dimensionless form of the following nonhomogeneous equation of motion:

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + ku = F_0 \cos\omega_0 t, \tag{40}$$

   where u is the displacement, m is the mass, F_0 and ω_0 are the driving force parameters, and t is the time. Additionally, determine the dimensions of damping coefficient b, stiffness k, driving force F_0 and driving force frequency ω_0.

[2]Link: `https://www.youtube.com/watch?v=P_YCvTabMO4`