Lecture 2: 1-D problems, linear analysis, bifurcation, bifurcation diagram

Contents

1 Introduction

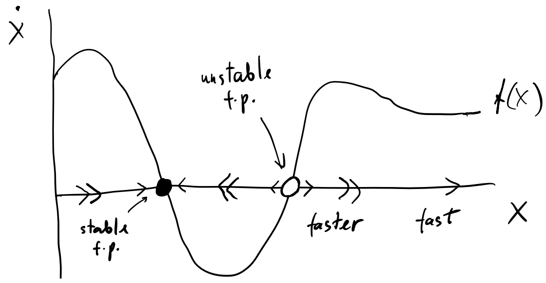

Figure 1: Phase portrait of a 1-D problem or a first-order problem. Fixed points are shown with the filled and unfilled bullets. Arrowheads show the direction of flow. The 1-D flow takes place on the x-axis.

We continue with the one-dimensional problems or the problems of flow on a line given in the following form:

$$\dot{x} = f(x), \tag{1}$$

where function $f(x)$ can be linear or nonlinear. Figure 1 shows the phase portrait of a 1-D problem. At times it is beneficial to simplify your problems in order to analyse them. This is especially true in case of problems that are described by nonlinear differential equations. **Linearisation** is a tool for analysing the dynamics of nonlinear systems.

2 Linearisation of 1-D systems

2.1 Linearisation about fixed point x^* and linear stability analysis of x^*

In mathematics, linearisation is finding the linear approximation to a function at a given point. The linear approximation of a function is the first order Taylor expansion around the point of interest. In the study of dynamical systems, linearisation is a method for assessing the local stability and type of a fixed point of a system of nonlinear differential equations or discrete dynamical systems.

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Linearisation of 1-D systems

The one-dimensional system is given by

$$\dot{x} = f(x). \tag{1}$$

The dynamics close to fixed point x^* can be expressed as follows

$$x(t) = x^* + \eta(t), \tag{2}$$

where $|\eta| \ll 1$ is a small perturbation. The behaviour and change of solution x over time thus is

$$\dot{x} = (x^* + \eta)^{\cdot} = \dot{\eta}. \tag{3}$$

At the same time (1) holds. This mans that the dynamics of small perturbations is the following

$$\dot{\eta} = f(x) = f(x^* + \eta). \tag{4}$$

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Linearisation of 1-D systems

Taylor series expansion about x^* of (4) results in

$$\dot{\eta} = f(x) = f(x^* + \eta) \tag{5}$$

$$= f(x^*) + \frac{f'(x^*)}{1!}(x^* + \eta - x^*) + \frac{f''(x^*)}{2!}(x^* + \eta - x^*)^2 + \ldots \tag{6}$$

$$= f'(x^*)\eta + \underbrace{\frac{f''(x^*)}{2!}\eta^2 + \ldots}_{\text{higher order terms, } O(\eta^2)} \tag{7}$$

$$\approx f'(x^*)\eta. \tag{8}$$

If $f'(x^*) \neq 0$, then the term $|f'(x^*)\eta| \gg \left|\frac{f''(x^*)}{2!}\eta^2\right|$. Neglecting $O(\eta^2)$ yields the linearisation of the system about fixed point x^*

$$\dot{\eta} = s\eta, \tag{9}$$

where $s = f'(x^*)$ is simply the slope of function $f(x)$ evaluated at x^*.

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Let's examine the dynamics of Eq. (1) close to its fixed point or points x^*. We assume the solution is in the following form:

$$x(t) = x^* + \eta(t), \tag{2}$$

where $|\eta| \ll 1$ is a small perturbation. The behaviour and change of solution x over time thus is

$$\dot{x} = (x^* + \eta)^{\cdot} = \dot{\eta}. \tag{3}$$

At the same time it holds that $\dot{x} = f(x)$. The combination of these results gives

$$\dot{\eta} = f(x) = f(x^* + \eta) = \begin{bmatrix}\text{Taylor ser.}\\ \text{expansion}\\ \text{about } x^*\end{bmatrix} = f(x^*) + \frac{f'(x^*)}{1!}(x^* + \eta - x^*) + \frac{f''(x^*)}{2!}(x^* + \eta - x^*)^2 + \cdots =$$

$$= \big[f(x^*) = 0\big] = f'(x^*)\eta + \underbrace{\frac{f''(x^*)}{2!}\eta^2 + \ldots}_{\text{higher order terms, } O(\eta^2)} \approx f'(x^*)\eta. \quad (4)$$

If $f'(x^*) \neq 0$, then the term $|f'(x^*)\eta| \gg \left|\frac{f''(x^*)}{2!}\eta^2\right|$. Neglecting $O(\eta^2)$ yields the linearisation of the system about fixed point x^*

$$\dot{\eta} = s\eta, \quad (5)$$

where $s = f'(x^*)$ is simply the slope of function $f(x)$ evaluated at x^*.

The linearised form (5) of the original system (1) is a familiar equation to us. Solution of Eq. (5) has the form

$$\eta(t) = \mathrm{e}^{C}\mathrm{e}^{st} = \eta_0\,\mathrm{e}^{st}, \quad (6)$$

where C is the constant of integration and η_0 is a suitable initial amplitude. The stability of this solution based on its behaviour is the following:

- If $s > 0$, then the solution is exponentially growing (exploding). We say that the solution dynamics and corresponding fixed point are **unstable**.
- If $s < 0$, then the solution is exponentially decaying. Asymptotically approaching a *stable* value ($\eta(t) \to 0$ for $t \to \infty$). We conclude that the solution dynamics and corresponding fixed point are **stable**.

This behaviour is also clearly seen on the example phase portrait shown in Fig. 2. The linearised phase portraits (shown in red) preserve the characteristics of the original flow (shown in black) in the close proximity of their respective fixed point x_1^* or x_2^*.

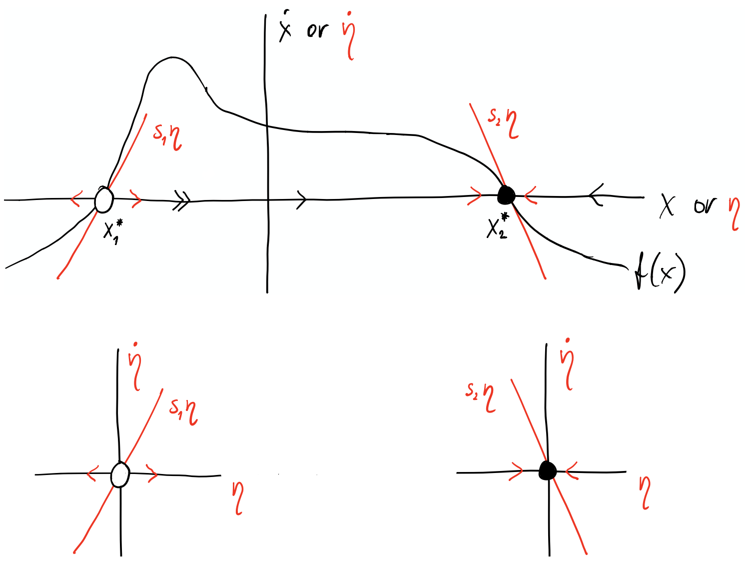


Figure 2: (Top) Phase portraits of the original flow described by Eq. (1) and its linearised counterparts given by Eq. (5). Linearisation is performed about fixed points x_1^* and x_2^*. Linearised solutions are shifted to coincide with the position of the fixed point. Linearised phase portraits are similar to the original flow only in close proximity to their respective fixed points. (Bottom) Aforementioned linearised solutions shown on their respective phase portraits.

If $s = f'(x^*) = 0$ then no information from linearisation can be obtained. The determination of stability type needs further analysis, i.e. additional terms in the Taylor series expansion (4) need to be considered.

2.2 Examples of cases where $f'(x^*) = 0$

Consider the following systems

$$\dot{x} = x^2, \tag{7}$$

$$\dot{x} = -x^2, \tag{8}$$

$$\dot{x} = x^3, \tag{9}$$

$$\dot{x} = -x^3. \tag{10}$$

Systems (7)–(10) share the same fixed point $x^* = 0$ and have different stability types, this can be clearly seen in Fig. 3. Systems (7) and (8) feature a new type of fixed point—the **half-stable fixed point**. The origin and nature of the half-stable fixed point will become clear below (Sec. 6.1).

Systems (9) and (10) feature **algebraic decay** $x(t) \sim t^{-\delta}$ near the fixed point (see Slide 5), as opposed to a more common **exponential decay** $x(t) \sim \mathrm{e}^{-\gamma t}$, where γ is constant. We will revisit this idea in future lectures (algebraic decay may be a possible source of *confusion* in analysis of 2-D systems).

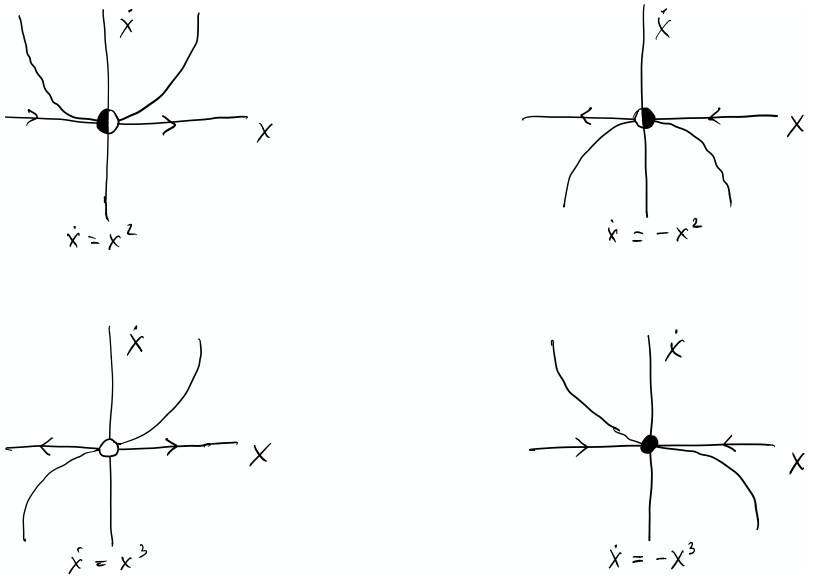
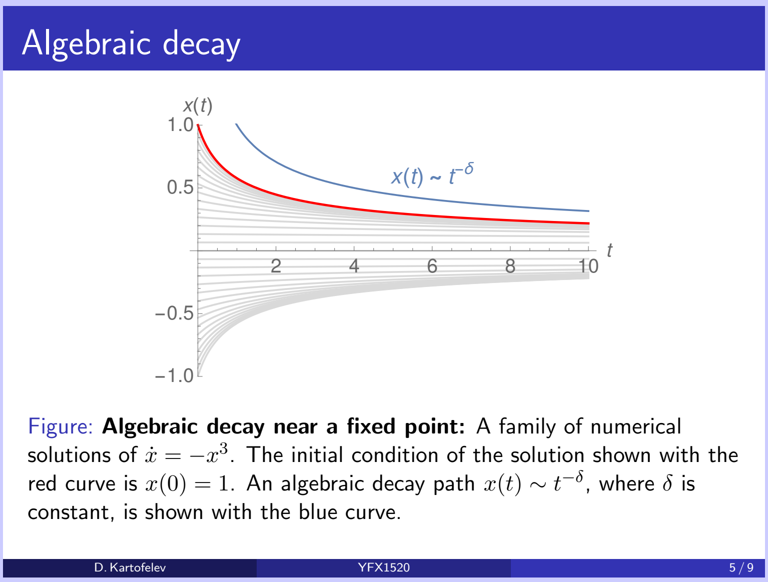


Figure 3: Phase portraits of Eqs. (7)–(10). Upper row of the graphs feature a new type of fixed point—the **half-stable fixed point** that is stable from one side but not on the other.

SLIDE: 5



3 Example of linearisation and linear stability analysis

3.1 Linearisation of the logistic equation

The logistic equation has the form

$$\dot{x} = f(x) = rx\left(1 - \frac{x}{K}\right), \tag{11}$$

where $x \geq 0$ is the population size, $r > 0$ and $K > 0$ are the system parameters (see Lecture 1). Fixed points are the following:

$$\dot{x} = 0 \;\Rightarrow\; rx^*\left(1 - \frac{x^*}{K}\right) = 0 \;\Rightarrow\; \begin{cases} x_1^* = 0, \\ x_2^* = K. \end{cases} \tag{12}$$

In order to obtain linearisation in the form (5), we calculate the slope of the right hand side of (11) and evaluate it at the fixed points $x_{1,2}^*$. The slope is

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[rx\left(1 - \frac{x}{K}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left(rx - \frac{r}{K}x^2\right) = r - \frac{2r}{K}x. \tag{13}$$

Evaluation of the obtained slope at x^* results in the following linearised system

$$\dot{\eta} = f'(x^*)\eta = \left(r - \frac{2r}{K}x\right)\Big|_{x=x^*}\eta. \tag{14}$$

In the case $x_1^* = 0$ we have

$$\dot{\eta} = \left(r - \frac{2r}{K}x\right)\Big|_{x=x_1^*=0}\eta = r\eta, \tag{15}$$

and in the case $x_2^* = K$ we have

$$\dot{\eta} = \left(r - \frac{2r}{K}x\right)\Big|_{x=x_2^*=K}\eta = (r - 2r)\eta = -r\eta. \tag{16}$$

The phase portrait shown in Fig. 4 shows the obtained linearised systems using the red graphs.

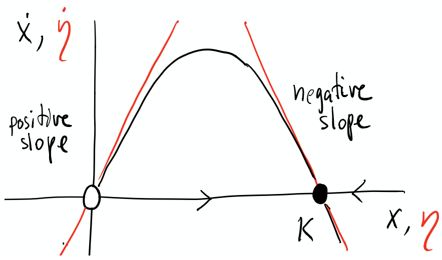


Figure 4: Stability of the fixed points is indicated by the hollow and filled bullets. Linearised results are shown with the red lines. $\dot{\eta} = r\eta$ for $x_1^* = 0$ and $\dot{\eta} = -r(\eta - K)$ for $x_2^* = K$, here the graph $\dot{\eta} = -r\eta$ (16) is shifted to coincide with the actual position of the fixed point $x_2^* = K$.

3.2 Linear stability analysis of the fixed points

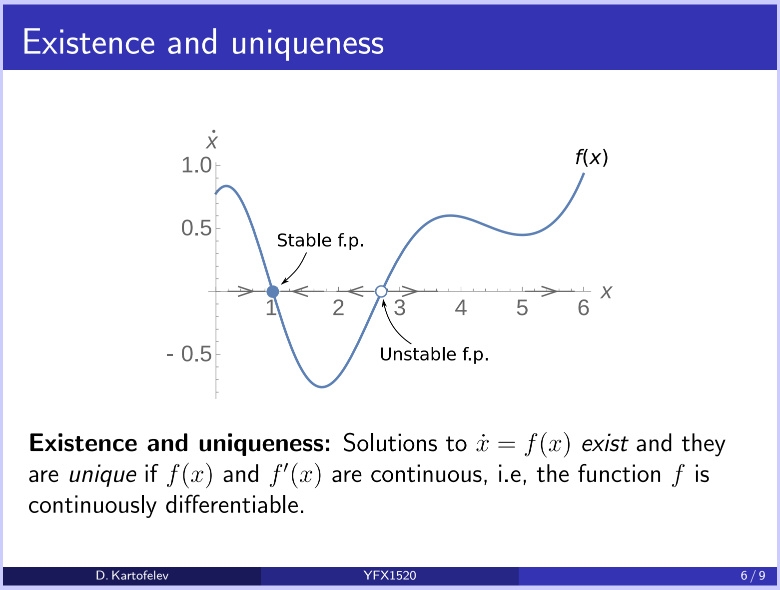
In order to evaluate the stability of the above fixed points we need to evaluate the signs of slopes $f'(x^*)$.

- For $x_1^* = 0$ we get: $f'(x^*) = \left(r - \frac{2r}{K}x\right)\big|_{x=x_1^*=0} = r$. The fixed point is **unstable** because the sign of the slope is **positive**.
- For $x_2^* = K$ we get: $f'(x^*) = \left(r - \frac{2r}{K}x\right)\big|_{x=x_2^*=K} = -r$. The fixed point is **stable** because the sign of the slope is **negative**.

Figure 4 shows the clear relationship between the slopes of the linearised results and the stability of the corresponding fixed points.

4 Existence and uniqueness of solutions of 1-D systems

SLIDE: 6



Graphically speaking uniqueness implies that the graphs of solutions $x(t)$ for different initial conditions $x(0)$ do not cross each other, *cf.* Slide 5. A single initial condition $x(0)$ can not result in more than one solution. The solution **trajectories** can't have more than one future both in forward or backward time t.

5 Impossibility of oscillations in 1-D systems

The 1-D systems are limited in what they can describe. The possible behaviours of solution $x(t)$ as $t \to \infty$, for $\dot{x} = f(x)$ are:

(i) $x(t) \to \pm\infty$ as $t \to \infty$.

(ii) $x(t) \to x^*$ as $t \to \infty$.

The behaviours that are not possible include: oscillations, periodic and quasi-periodic solutions, chaos (last two defined/explained in future lectures). Two examples of impossible behaviours are shown in Fig. 5

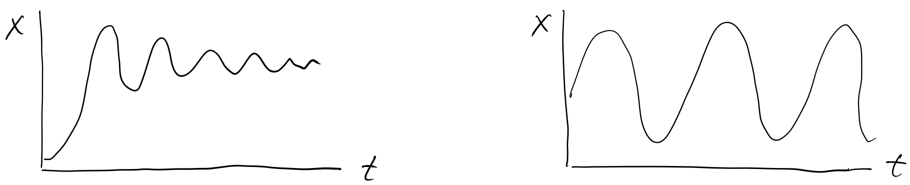


Figure 5: Examples of impossible behaviours in 1-D systems: (Left) oscillation with attenuation, (Right) harmonic oscillation.

Why is this true? Let's study Fig. 6 where the flow is always damped out at the stable fixed points for $t \to \infty$. By definition 1-D systems do not have inertial terms ($\ddot{x}$) this means that all forces are always balanced out by the damping, remember that $\dot{x}$ term is proportional to damping, see Eq. (1).

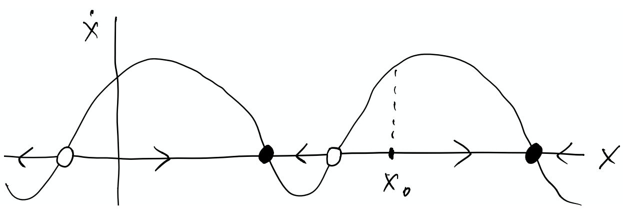
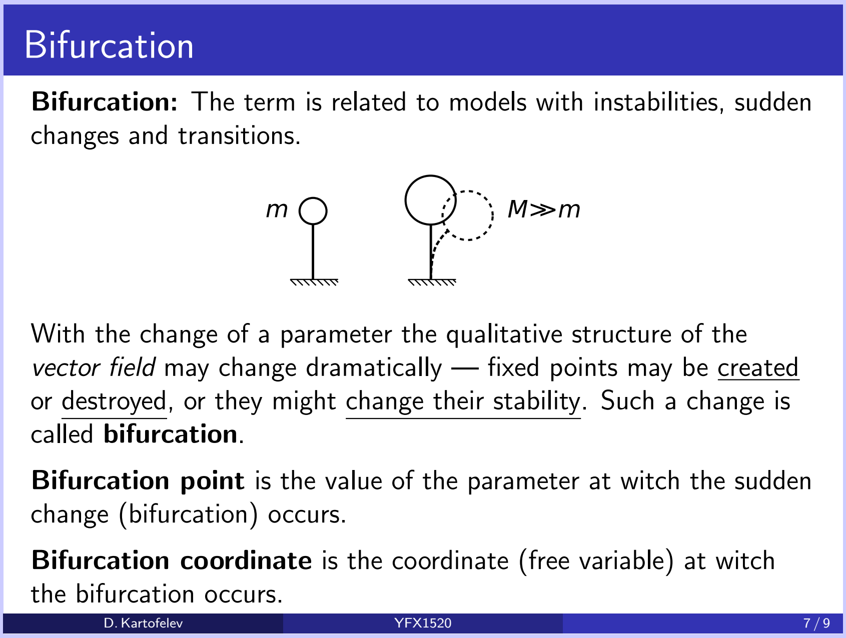


Figure 6: Phase portrait of an arbitrary 1-D flow problem where $x_0 = x(0)$ is the initial condition.

6 Bifurcation

SLIDE: 7



Slide 7 gives the definition of bifurcation. In this section our goal is to familiarise ourselves with a *selection* of basic bifurcation dynamics that occur in 1-D systems. They are:

- Saddle-node bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
  - Supercritical pitchfork bifurcation
  - Subcritical pitchfork bifurcation

6.1 Saddle-node bifurcation

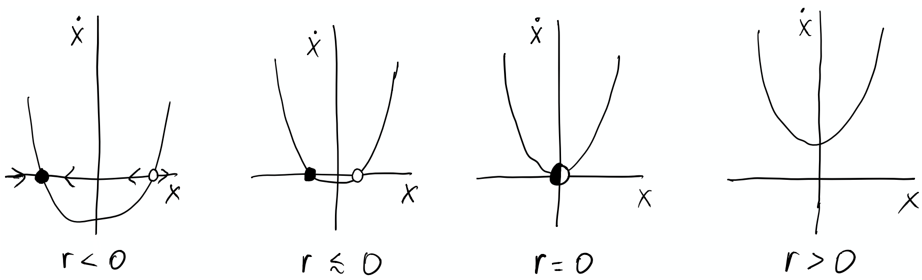


Figure 7: Phase portraits of Eq. (17) undergoing saddle-node bifurcation.

Basic mechanism for creation or destruction of fixed points. The *standard* example is

$$\dot{x} = r + x^2, \tag{17}$$

where r is the **control parameter**. Figure 7 shows phase portraits corresponding to different values of r. Fixed points exist only for $r \leq 0$ and they are given by

$$\dot{x} = 0 \quad \Rightarrow \quad r + x^{*2} = 0 \quad \Rightarrow \quad x^* = \pm \mathrm{i}\sqrt{r}, \quad r \leq 0, \tag{18}$$

where i is the imaginary unit. The **bifurcation point** value $r = 0$ and it occurs at spatial point $x = 0$, we write $(x, r) = (0, 0)$. This example also shows the origin of the mysterious **half-stable fixed point** present in Eqs. (7) and (8). In those examples we simply caught the system at the moment of bifurcation.

A similar general behaviour, i.e. bifurcation dynamics, happens for the system with inverted parabola

$$\dot{x} = r - x^2. \tag{19}$$

Here, the fixed point exists only for $r \geq 0$, their stability is the opposite, and they are

$$\dot{x} = 0 \quad \Rightarrow \quad r - x^{*2} = 0 \quad \Rightarrow \quad x^* = \pm\sqrt{r}, \quad r \geq 0. \tag{20}$$

6.1.1 Bifurcation diagram

Bifurcation diagram is obtained by plotting the graph of fixed point x^* as a function of control parameter r, analytically expressed by (18). The bifurcation diagram of Eq. (17) is shown in Fig. 8. Compare Fig. 8 to the corresponding Fig. 7.

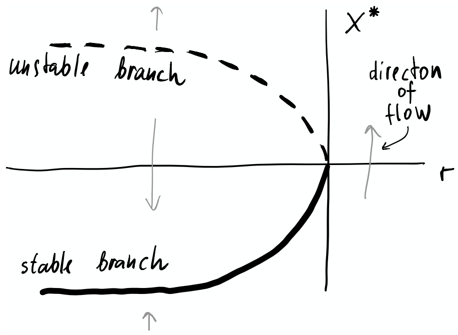


Figure 8: Bifurcation diagram of the saddle-node bifurcation. The unstable branch is shown with the dashed bold curve and the stable branch with the continuous bold curve.

The following numerical file shows the quantitatively accurate phase portraits and bifurcation diagrams discussed in this section.

> NUMERICS: NB#1
> Bifurcation diagrams and 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

6.1.2 Not so clear example of the saddle-node bifurcation

Let's consider the following problem

$$\dot{x} = r + x - \ln(1 + x), \tag{21}$$

where r is the control parameter. It is impossible to determine the dynamics of the fixed points algebraically. Try it, and/or see the numerical file linked below.

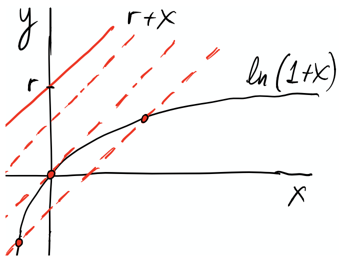
> NUMERICS: NB#1
> Bifurcation diagrams and 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).
>
> **Note:** The red error messages from Mathematica are left in intentionally. The symbolic calculation packages (CAS – computer algebra system) can not handle the problem using inverse functions.

We employ a graphical approach to find the fixed points, bifurcation point and their dependents on control parameter r. This means that we need to solve

$$\dot{x} = 0 \quad \Rightarrow \quad r + x - \ln(1 + x) = 0, \tag{22}$$

or equivalently

$$r + x = \ln(1 + x), \tag{23}$$



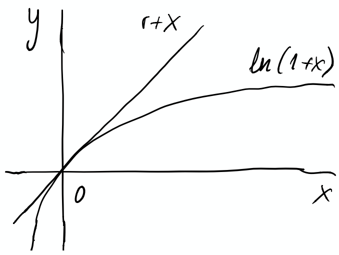


Figure 9: (Left) Behaviour of functions (24) as the value of r is changed. (Right) Bifurcation happening for the tangent intersection of the line and the curve given by (24).

for variable x. We can use an arbitrary axis y to plot the left hand side and the right hand side of Eq. (23)

$$y = r + x \quad \text{and} \quad y = \ln(1 + x). \tag{24}$$

Figure 9 shows the behaviour of the line and the curve given by (24) as the value of r is changed. The saddle-node bifurcation occurs when the tangent intersection takes place. We write and it must hold

$$\begin{cases} r + x = \ln(1 + x), \\ \dfrac{\mathrm{d}}{\mathrm{d}x}(r + x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\ln(1 + x). \end{cases} \tag{25}$$

As stated above the first equation in Sys. (25) is *hard* to solve algebraically, but the second one is easier. After taking the derivatives we get

$$1 = \frac{1}{1 + x} \quad \Rightarrow \quad x^* = 0. \tag{26}$$

The corresponding r value can be found from the first equation in Sys. (25)

$$(r + x)|_{x=x^*=0} = \ln(1 + x)|_{x=x^*=0}, \tag{27}$$

$$r = \ln 1 = 0. \tag{28}$$

Thus, the bifurcation coordinate and point $(x, r) = (0, 0)$. The computer analysis of this problem is linked below.

Numerics: nb#1

Bifurcation diagrams and 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

File content: A quantitatively precise behaviour of the dynamics shown in Fig. 9; the phase portrait and bifurcation diagram corresponding to Eq. (21). Figure 10 shows the resulting diagram.

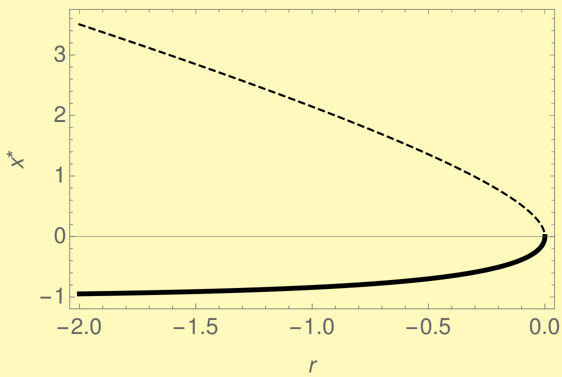


Figure 10: Bifurcation diagram of the problem given by (21). The stable branch is shown with the solid bold curve and the unstable branch is shown with the dashed curve.

6.1.3 Normal form

Let's take a closer look at the previous result. The series expansion of Eq. (21) is in the form

$$\dot{x} = r + x - \ln(1+x) \approx \begin{bmatrix} \text{Maclaurin ser.} \\ \text{expansion of} \\ \ln(1+x) \\ \text{about } x^* = 0 \end{bmatrix} \approx r + x - \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)}_{\text{Maclaurin expansion}} \approx r + \frac{x^2}{2} + O(x^3). \quad (29)$$

This result is similar to the example given by Eq. (17). Surely, after neglecting $O(x^3)$ terms, the qualitative dynamics of this system will be *exactly* the same. Many systems can be reduced to the form (17) or (19) near their respective fixed points. The **normal form** of the saddle-node bifurcation is thus given by

$$\boxed{\dot{x} = r \pm x^2.} \quad (30)$$

All systems that can be reduced to this form share the above properties.

6.2 Transcritical bifurcation

The normal form of the transcritical bifurcation is given by

$$\boxed{\dot{x} = rx \pm x^2,} \quad (31)$$

where r is the control parameter. Let's study the case

$$\dot{x} = rx - x^2 = x(r - x). \quad (32)$$

Fixed points for this system are

$$\begin{cases} x_1^* = 0, & \forall r, \\ x_2^* = r. \end{cases} \quad (33)$$

Figures 11 and 12 show the phase portraits for the varied values of r and the corresponding bifurcation diagram, respectively. **Note:** $x^* = 0$ can't be destroyed but its stability can be changed. From Fig. 12 it is clear that the bifurcation coordinate and point $(x, r) = (0, 0)$.

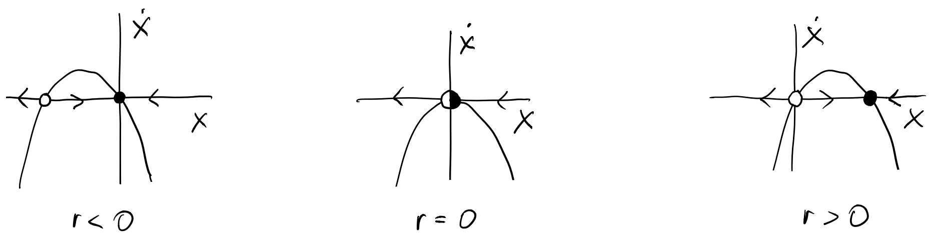


Figure 11: Phase portraits of Eq. (32) for different values of r.

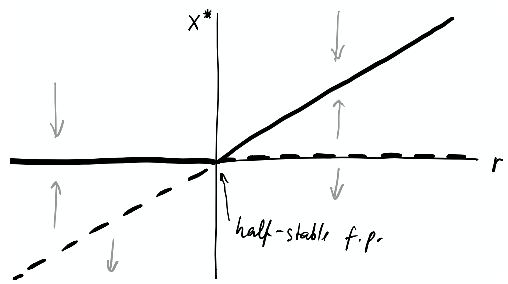


Figure 12: Bifurcation diagram of Eq. (32). The stable branches are shown with the bold continuous lines and the unstable branches are shown with the dashed bold lines.

These result can be compared against the result found using a computer.

NUMERICS: NB#1
Bifurcation diagrams and 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

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Let's analyse algebraically the stability of the bifurcation diagram branches shown in Fig. 12. In order to determine the stability of the fixed points one needs to analyse the slopes of phase portrait curves at said fixed points x^*.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(rx - x^2) = r - 2x \tag{34}$$

In case of $x_1^* = 0$

$$f'(x_1^*) = f'(0) = r, \quad x_1^* \text{ is } \begin{cases} \text{stable for } r < 0, \\ \text{unstable for } r > 0. \end{cases} \tag{35}$$

In case of $x_2^* = r$

$$f'(x_2^*) = f'(r) = -r, \quad x_2^* \text{ is } \begin{cases} \text{stable for } r > 0, \\ \text{unstable for } r < 0. \end{cases} \tag{36}$$

Indeed, these result agrees with the graphical and numerical results presented above.

——— *Skip if needed:* stop ———

6.3 Pitchfork bifurcation

The pitchfork bifurcation occurs in systems with symmetry. It involves mergers or splitting apart of fixed points and changes in their stabilities. Normal form is given by

$$\boxed{\dot{x} = rx \pm x^3,} \tag{37}$$

where r is the control parameter.

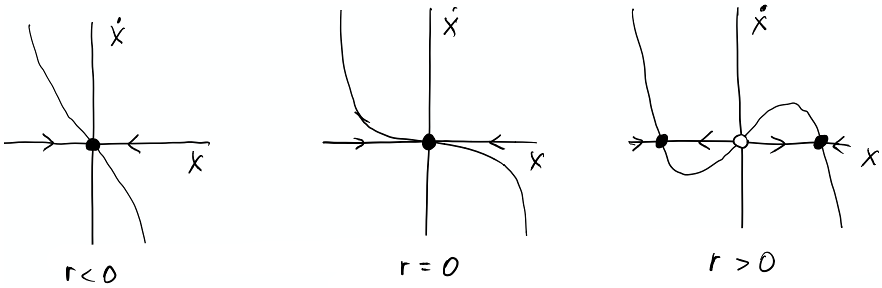

Figure 13: Phase portrait of Eq. (38) for different values of r. In the case $r = 0$ solutions near the fixed point $x = x^* = 0$ decay algebraically.

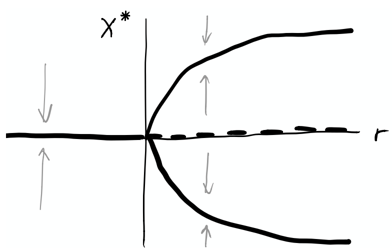

Figure 14: Bifurcation diagram. The stable branches are shown with the bold continuous curves and a lines and the unstable branch is shown with bold dashed line.

6.3.1 Supercritical pitchfork bifurcation

Let's study the case

$$\dot{x} = rx - x^3. \tag{38}$$

Fixed points for this system are

$$\begin{cases} x_1^* = 0, \quad \forall r, \\ x_2^* = \pm\sqrt{r}, \quad r \geq 0. \end{cases} \tag{39}$$

Figures 13 and 14 show the phase portraits for varied values of r and the corresponding bifurcation diagram, respectively. From Fig. 14 it is clear that the bifurcation coordinate and point $(x, r) = (0, 0)$. The above result can be compared against the result found using a computer.

> NUMERICS: NB#1
> Bifurcation diagrams and 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

6.3.2 Subcritical pitchfork bifurcation

Let's consider the other case of normal form (37) where

$$\dot{x} = rx + x^3. \tag{40}$$

Fixed points for this system are the same as for the previous case but with different stability types. The fixed point are

$$\begin{cases} x_1^* = 0, \quad \forall r, \\ x_2^* = \pm \mathrm{i}\sqrt{r}, \quad r < 0, \end{cases} \tag{41}$$

where i is the imaginary unit.

Figures 15 and 16 show the phase portraits for varied values of r and the corresponding bifurcation diagram, respectively. From Fig. 16 it is clear that the bifurcation coordinate and point $(x, r) = (0, 0)$.

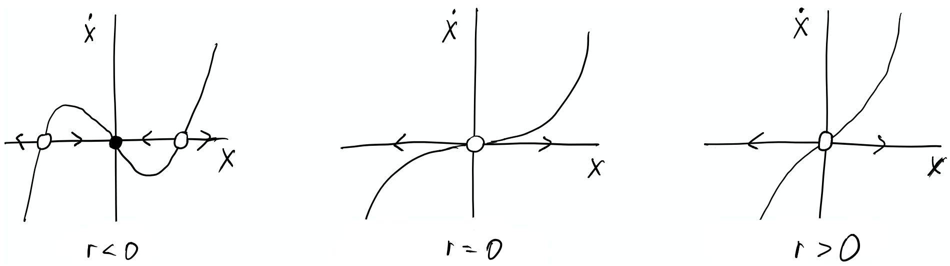

Figure 15: Phase portrait of Eq. (40) for different values of r. In the case $r = 0$ the solutions near the fixed point $x^* = 0$ decay algebraically.

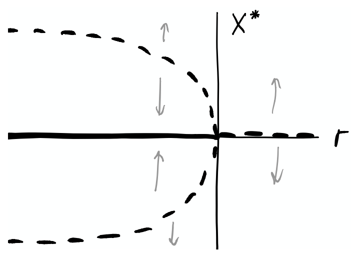

Figure 16: Bifurcation diagram. The stable branch is shown with the continuous bold line and the unstable branches are shown with the dashed bold curves and a line.

The obtained result can be compared against the result found using a computer.

NUMERICS: NB#1

Bifurcation diagrams and 1-D systems: saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation (supercritical), pitchfork bifurcation (subcritical).

Revision questions

1. What does linearisation of a nonlinear system imply?
2. Linearise the following 1-D system
$$\dot{x} = x^3 - x \tag{42}$$
3. What is bifurcation?
4. What is bifurcation diagram?
5. What is saddle-node bifurcation?
6. What is transcritical bifurcation?
7. What is pitchfork bifurcation?
8. What is supercritical pitchfork bifurcation?
9. What is subcritical pitchfork bifurcation?
10. What is normal form in the context of bifurcations?
11. Are oscillation possible in 1-D systems?
12. Why are oscillations impossible in 1-D systems?
13. What does uniqueness of solutions imply in the context of phase space trajectories?