Noname manuscript No.
(will be inserted by the editor)

On the wave dispersion in microstructured solids

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Received: date / Accepted: date

Abstract In this paper, elastic wave propagation in a one-dimensional micromorphic medium characterized by two internal variables is investigated. The evolution equations are deduced following two different approaches, namely using: i) the balance of linear momentum and the Clausius-Duhem inequality and, ii) an assumed Lagrangian functional (including a gyroscopic coupling) together with a variational principle. The dispersion relation is obtained and the possibility of the emerging band gaps is shown in such microstructured materials. Some numerical simulations are also performed in order to highlight the dispersive nature of the material under study.

Keywords Micromorphic media · wave propagation · internal variables

1 Introduction

Elastic wave dispersion is very often an indicator of the presence of a microstructure in a solid. The study of the wave dispersion in bodies with regular microstructures goes back to Newton and Euler (see [15] for the historical background). This line of research resulted in crystal lattice theory [13], and is come back in fashion in recent years due to the possibility of manufacturing materials with a microstructure which is very finely controlled by means of computer-aided methods [49,25, 22,58].

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Irregular or natural microstructures require different treatment due to their stochastic character. Among the proposed approaches, homogenized, generalized, and microstructural descriptions are able to take into account a microstructural influence on the macro-motion. Classical homogenization uses "effective" (averaged) quantities without changes in governing equations [52,54]. The main task here is to find the appropriate values of "effective" material properties. Wave dispersion does not appear here.

Asymptotic homogenization is more sophisticated. Here, governing equations include additional high-order terms (general reference on higher-order continua are [24,2,10]) which provide the wave dispersion [17,33,4,38]. The advantage of the asymptotic homogenization is the possibility of the prediction of additional material parameters, althiugh they usually require more advanced mathematical techniques [14,48]. Computational homogenization [36,57,41,51] includes multiscale treatment. Recently, it was demonstrated that computational homogenization can result in the micromorphic microelasticity [11].

The micromorphic microelasticity theory [50,28] postulates the existence of a micromotion in addition to a macromotion. This theory predicts both acoustical and optical branches of a dispersion curve, which is not the case for more simplified gradient models [6,34]. The disadvantage of the micromorphic microelasticity is in a certain amount of additional material parameters depending on the complexity of the microstructure under consideration. Therefore, some kind of the reduction of the micromorphic theory can be useful [53].

The micromorphic microelasticity theory is a particular case of the general microstructural approach [16,42,27,3,35], which introduces a set of directors for the description of microstructure. This approach can be useful in deterministic as well as in stochastic contexts, as exemplified by the well-known Ising and Potts models [55,19,40]. However, evolution equations for these directors, as well as boundary conditions for them, are not straightforward.

Another particular case of the general microstructural approach is presented by theory of internal variables [47,45]. The recent extension of this theory recover the micromorphic microelasticity theory [9] and provides the evolution equations for internal variables as the consequence of the second law of thermodynamics [59]. It was shown that the dual internal variables approach unifies all known models for the elastic wave dispersion [8]. However, this was demonstrated on particular examples, while it is much more difficult to analyze the general case.

In what follows, the quadratic dependence of free energy on dual internal variables is used in the one-dimensional setting after the reminder of the structure of evolution equations for these variables. Then, the complete dispersion relation is derived and studied. Numerical simulations of pulse propagation through a microstructured solid confirm the possibility of stop bands predicted theoretically.

2 Theoretical background

A motion of a body is considered as a time-parametrized sequence of mappings χ between the reference configuration and the actual configuration: $\mathbf{x} = \chi(\mathbf{X}, t)$, where t is time, $\mathbf{X}$ represents the position of a material point in the reference configuration, and $\mathbf{x}$ is its position in the actual configuration. The deformation

gradient is defined by

$$\mathbf{F} = \partial\chi/\partial\mathbf{X}|_t = \nabla_R\chi. \tag{1}$$

If the constitutive relation for free energy has the form $W = \overline{W}(\mathbf{F}, ..., \mathbf{X}, t)$, then the first Piola-Kirchhoff stress tensor $\mathbf{T}$ is defined by

$$\mathbf{T} = \frac{\partial\overline{W}}{\partial\mathbf{F}}. \tag{2}$$

In the so-called **Piola-Kirchhoff formulation** local balance laws of mass, physical (linear) momentum, and energy for sufficiently smooth fields at any regular material point $\mathbf{X}$ in the body read (cf. [44]):

$$\left.\frac{\partial\rho_0}{\partial t}\right|_{\mathbf{X}} = 0, \tag{3}$$

$$\left.\frac{\partial(\rho_0\mathbf{v})}{\partial t}\right|_{\mathbf{X}} - Div_R\mathbf{T} = \mathbf{f}_0, \tag{4}$$

$$\left.\frac{\partial(K+E)}{\partial t}\right|_{\mathbf{X}} - \nabla_R\cdot(\mathbf{T}\cdot\mathbf{v} - \mathbf{Q}) = \mathbf{f}_0\cdot\mathbf{v}, \tag{5}$$

where ρ_0 is the mass density in the reference configuration, $\mathbf{v} = \partial\chi/\partial t|_X$ is the physical velocity, $\mathbf{f}_0$ is a body force per unit reference volume, $K = \rho_0\mathbf{v}^2/2$ is the kinetic energy, E is the internal energy per unit reference volume, $\mathbf{Q}$ is the material heat flux, $d/dt = \partial/\partial t|_X$ or a superimposed dot denotes the material time derivative.

The second law of thermodynamics is written as [45, e.g.]

$$\left.\frac{\partial S}{\partial t}\right|_{\mathbf{X}} + \nabla_R\cdot\mathbf{S} \geq 0, \quad \mathbf{S} = (\mathbf{Q}/\theta) + \mathbf{K}, \tag{6}$$

where S is the entropy density per unit reference volume, θ is the absolute temperature, $\mathbf{S}$ is the entropy flux, and the "extra entropy flux" $\mathbf{K}$ vanishes in most cases, but this is not a basic requirement.

The canonical form of the energy conservation has the form [45]

$$\left.\frac{\partial(S\theta)}{\partial t}\right|_{\mathbf{X}} + \nabla_R\cdot\mathbf{Q} = h^{int}, \quad h^{int} := \mathbf{T}:\dot{\mathbf{F}} - \left.\frac{\partial\overline{W}}{\partial t}\right|_{\mathbf{X}}, \tag{7}$$

where the right-hand side of Eq. $(7)_1$ is formally an internal heat source.

Correspondingly, the canonical (material) momentum conservation equation is obtained as [45]

$$\left.\frac{\partial\mathbf{P}}{\partial t}\right|_{\mathbf{X}} - Div_R\mathbf{b} = \mathbf{f}^{int} + \mathbf{f}^{ext} + \mathbf{f}^{inh}, \tag{8}$$

where the *material momentum* $\mathbf{P}$, the material *Eshelby stress* $\mathbf{b}$, the material *inhomogeneity force* $\mathbf{f}^{inh}$, the material *external* (or body) force $\mathbf{f}^{ext}$, and the material *internal* force $\mathbf{f}^{int}$ are defined by

$$\mathbf{P} := -\rho_0\mathbf{v}\cdot\mathbf{F}, \quad \mathbf{b} = -\left(L\mathbf{I}_R + \mathbf{T}\cdot\mathbf{F}\right), \quad L = K - \overline{W}, \tag{9}$$

$$\mathbf{f}^{inh} := \left.\frac{\partial L}{\partial\mathbf{X}}\right|_{expl} \equiv \left.\frac{\partial L}{\partial\mathbf{X}}\right|_{fixed\ fields} = \left(\frac{1}{2}\mathbf{v}^2\right)\nabla_R\rho_0 - \left.\frac{\partial\overline{W}}{\partial\mathbf{X}}\right|_{expl}, \tag{10}$$

$$\mathbf{f}^{ext} := -\mathbf{f}_0\cdot\mathbf{F}, \quad \mathbf{f}^{int} = \mathbf{T}:(\nabla_R\mathbf{F})^T - \nabla_R\overline{W}|_{impl}. \tag{11}$$

Here the subscript notations *expl* and *impl* mean, respectively, the material gradient keeping the fields fixed (and thus extracting the explicit dependence on $\mathbf{X}$), and taking the material gradient only through the fields present in the function, $\mathbf{I}_R$ is the unit matrix in the reference configuration.

3 Single internal variable. One-dimensional case

Up to now the microstructure was not specified. In the framework of the phenomenological continuum theory it is assumed that the influence of the microstructure on the overall macroscopic behavior can be taken into account by the introduction of an internal variable φ which we associate with the integral distributed effect of the microstructure. The idea of internal variables has a long history [46]. The theory of a single internal variable is well established [45]. It is instructive to remind the reader how it works on the example of one-dimensional case.

The free energy W is specified as the general sufficiently regular function of the strain, temperature, the internal variable, and its space gradient [45]

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x). \tag{12}$$

The corresponding *equations of state* define not only the stress and entropy, but also the "microstress" η and interactive "force" τ

$$\sigma = \frac{\partial \overline{W}}{\partial u_x}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi} \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}. \tag{13}$$

Following the scheme originally developed by [43] for materials with *diffusive* dissipative processes described by means of internal variables of state, we chose the non-zero extra entropy flux K in the form

$$K = -\theta^{-1}\eta\dot{\varphi}. \tag{14}$$

In this case, the "internal" material force and heat source each split in two terms according to

$$f^{int} = f^{th} + \widetilde{f}^{intr}, \quad h^{int} = h^{th} + \widetilde{h}^{intr}, \tag{15}$$

where the *thermal sources* and the *"intrinsic" sources* are given by [45]

$$\begin{aligned} f^{th} &:= S\frac{\partial}{\partial x}\theta, \quad h^{th} := S\dot{\theta}, \\ \widetilde{f}^{intr} &:= \widetilde{\tau}\frac{\partial \varphi}{\partial x}, \quad \widetilde{h}^{intr} := \widetilde{\tau}\dot{\varphi}, \end{aligned} \tag{16}$$

so that we have the following consistent canonical equations of momentum and energy:

$$\frac{\partial P}{\partial t} - \frac{\partial \widetilde{b}}{\partial x} = f^{th} + \widetilde{f}^{intr}, \tag{17}$$

$$\frac{\partial (S\theta)}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x} = h^{th} + \widetilde{h}^{intr}, \tag{18}$$

with dissipation

$$\Phi = \widetilde{h}^{intr} - \left(\frac{Q - \eta\dot{\varphi}}{\theta}\right)\frac{\partial \theta}{\partial x} \geq 0, \tag{19}$$

where the new definitions are introduced [45]:

$$\begin{aligned}\widetilde{\tau} &\equiv -\frac{\delta\overline{W}}{\delta\varphi} := -\left(\frac{\partial\overline{W}}{\partial\varphi} - \frac{\partial}{\partial x}\left(\frac{\partial\overline{W}}{\partial\varphi_x}\right)\right) = \tau - \eta_x,\\ \widetilde{b} &= -\left(\rho_0 v^2/2 - W + \sigma u_x - \eta\varphi_x\right).\end{aligned} \tag{20}$$

In the isothermal case, dissipation inequality (19) is automatically satisfied if

$$\widetilde{\tau} = k\dot{\varphi}$$

with $k \geq 0$ since

$$\Phi = k\dot{\varphi}^2 \geq 0. \tag{21}$$

The fully non-dissipative case corresponds to $k = 0$.

To be more specific, we consider a simple free energy dependence as a quadratic function

$$\overline{W} = \frac{\rho_0 c^2}{2}u_x^2 + A\varphi_x u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2, \tag{22}$$

where c is the elastic wave speed in the medium without microstructure, and constant parameters A, B, and C depend on the material. The corresponding stresses are calculated as follows:

$$\sigma = \frac{\partial\overline{W}}{\partial u_x} = \rho_0 c^2 u_x + A\varphi_x, \quad \eta = -\frac{\partial\overline{W}}{\partial\varphi_x} = -Au_x - C\varphi_x, \tag{23}$$

and the interactive internal force τ is, respectively,

$$\tau = -\frac{\partial\overline{W}}{\partial\varphi} = -B\varphi. \tag{24}$$

Consequently, the balance of linear momentum is rewritten as

$$u_{tt} = c^2 u_{xx} + \frac{A}{\rho_0}\varphi_{xx}, \tag{25}$$

and the evolution equation for the internal variable in the fully non-dissipative case (with $k = 0$) reduces to

$$C\varphi_{xx} - Au_{xx} - B\varphi = 0. \tag{26}$$

Evaluating the second space derivative of the internal variable from the last equation

$$\varphi_{xx} = \frac{C}{B}\varphi_{xxxx} - \frac{A}{B}u_{xxxx}, \tag{27}$$

and its fourth space derivative from eqn. (25)

$$\frac{A}{\rho_0}\varphi_{xxxx} = \left(u_{tt} - c^2 u_{xx}\right)_{xx}, \tag{28}$$

we will have, inserting the results into the balance of linear momentum

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{A^2}{\rho_0 B}u_{xxxx}. \tag{29}$$

Equation (29) is the most general model for the dispersive wave motion provided by the single internal variable theory [8, cf.].

If coefficient C vanishes then we arrive at the strain gradient model

$$\varphi = \frac{A}{B} u_{xx}, \tag{30}$$

which results in the equation of motion of the form

$$\rho u_{tt} = \rho c^2 u_{xx} + \frac{A^2}{B} u_{xxxx}. \tag{31}$$

As it is shown in [8], dispersive wave equation (29) does not describe the optical branch of dispersion curve. To go further, we need to introduce one more internal variable following [59].

4 Dual internal variables. One-dimensional case

For a more general description of wave dispersion we need to use the extension of the internal variable theory onto the case of dual internal variables [59]. Let us consider the free energy W as a (sufficiently smooth) function of two internal variables φ, ψ and their space derivatives

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x, \psi, \psi_x). \tag{32}$$

In this case the equations of state are given by

$$\sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad S := -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \tag{33}$$

$$\xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x}. \tag{34}$$

The non-zero extra entropy flux is included into consideration similarly to the case of single internal variable

$$K = -\theta^{-1} \eta \varphi_t - \theta^{-1} \zeta \xi_t. \tag{35}$$

The canonical equations of momentum and energy keep their form

$$\frac{\partial P}{\partial t} - \frac{\partial \widetilde{b}}{\partial x} = f^{th} + \widetilde{f}^{intr}, \tag{36}$$

$$\frac{\partial (S\theta)}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x} = h^{th} + \widetilde{h}^{intr}, \tag{37}$$

with the modified Eshelby stress tensor

$$\widetilde{b} = -\left(\frac{1}{2} \rho v^2 - W + \sigma u_x - \eta \varphi_x - \zeta \psi_x \right) \tag{38}$$

and intrinsic source terms

$$\widetilde{f}^{intr} := \widetilde{\tau} \varphi_x + \widetilde{\xi} \psi_x, \quad \widetilde{h}^{intr} := \widetilde{\tau} \varphi_t + \widetilde{\xi} \psi_t. \tag{39}$$

In the above equations the following definitions are used

$$\widetilde{\tau} \equiv -\frac{\delta \overline{W}}{\delta \varphi} := -\left(\frac{\partial \overline{W}}{\partial \varphi} - \frac{\partial}{\partial x}\left(\frac{\partial \overline{W}}{\partial \varphi_x}\right)\right) = \tau - \eta_x, \tag{40}$$

$$\widetilde{\xi} \equiv -\frac{\delta \overline{W}}{\delta \psi} := -\left(\frac{\partial \overline{W}}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial \overline{W}}{\partial \psi_x}\right)\right) = \xi - \zeta_x, \tag{41}$$

$$\widetilde{S} = \theta^{-1}\widetilde{Q}, \quad \widetilde{Q} = Q - \eta\dot{\varphi} - \zeta\dot{\psi}, \tag{42}$$

which are similar to those in the case of single internal variable.

The corresponding dissipation is determined by

$$\Phi = \widetilde{h}^{intr} - \widetilde{S}\theta_x = \widetilde{\tau}\varphi_t + \widetilde{\xi}\psi_t - \widetilde{S}\theta_x \geq 0. \tag{43}$$

In the isothermal case, the dissipation inequality reduces to the intrinsic part depending only on internal variables

$$\Phi = \widetilde{h}^{intr} = \widetilde{\tau}\varphi_t + \widetilde{\xi}\psi_t = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \geq 0. \tag{44}$$

It is easy to see that the choice

$$\varphi_t = R(\xi - \zeta_x), \qquad \psi_t = -R(\tau - \eta_x), \tag{45}$$

where R is an appropriate constant, leads to zero dissipation. Therefore, the dissipation inequality (44) is satisfied automatically with the choice (45). The latter two evolution equations express the duality between internal variables: one internal variable is driven by another one and vice versa.

5 Microstructure description

Now we return to the description of a microstructure. The free energy function is constructed here as the sum of two similar contributions of internal variables

$$\begin{aligned} W = \frac{\rho_0 c^2}{2}u_x^2 + A_1\varphi_1 u_x + A_1'(\varphi_1)_x u_x + \frac{1}{2}B_1\varphi_1^2 + \frac{1}{2}C_1(\varphi_1)_x^2 \\ + A_2\varphi_2 u_x + A_2'(\varphi_2)_x u_x + \frac{1}{2}B_2\varphi_2^2 + \frac{1}{2}C_2(\varphi_2)_x^2. \end{aligned} \tag{46}$$

The corresponding stresses are determined as

$$\begin{aligned} \sigma &= \frac{\partial W}{\partial u_x} = \rho_0 c^2 u_x + A_1\varphi_1 + A_1'(\varphi_1)_x + A_2\varphi_2 + A_2'(\varphi_2)_x, \\ \eta_1 &= -\frac{\partial W}{\partial (\varphi_1)_x} = -C_1(\varphi_1)_x - A_1'u_x, \\ \eta_2 &= -\frac{\partial W}{\partial (\varphi_2)_x} = -C_2(\varphi_2)_x - A_2'u_x, \end{aligned} \tag{47}$$

as well as the interactive internal forces

$$\tau_1 = -\frac{\partial W}{\partial \varphi_1} = -A_1 u_x - B_1\varphi_1, \quad \tau_2 = -\frac{\partial W}{\partial \varphi_2} = -A_2 u_x - B_2\varphi_2. \tag{48}$$

Accordingly, the balance of linear momentum results in

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A_1(\varphi_1)_x + A_1'(\varphi_1)_{xx} + A_2(\varphi_2)_x + A_2'(\varphi_2)_{xx}, \tag{49}$$

and evolution equations for internal variables (45) have the form

$$(\varphi_1)_t = R(\tau_2 - (\eta_2)_x), \quad (\varphi_2)_t = -R(\tau_1 - (\eta_1)_x), \tag{50}$$

or, in terms of internal variables

$$(\varphi_1)_t = R(C_2(\varphi_2)_{xx} + A_2' u_{xx} - A_2 u_x - B_2\varphi_2), \tag{51}$$

$$(\varphi_2)_t = -R(C_1(\varphi_1)_{xx} + A_1' u_{xx} - A_1 u_x - B_1\varphi_1). \tag{52}$$

The obtained evolution equations are coupled with the balance of linear momentum (49).

6 Deduction of evolution equations via a variational principle

The variational principle can be alternatively used to derive the evolution equations derived in the previous sections, by using the D'Alembertian postulation instead of the postulation based on balance equations and Clausius-Duhem inequality. This methodological choice can be regarded as preferable by somebody for its conciseness and efficacity (see e.g. [23,30,31,32,29,7,20,21]). In order to do this, following the standard procedure due to Hamilton [60], we introduce a Lagrangian density function composed by kinetic and potential energies as follows

$$T = \frac{1}{2}\dot{\boldsymbol{d}} \cdot M\dot{\boldsymbol{d}} + G, \tag{53}$$

$$W = \frac{1}{2}\boldsymbol{d} \cdot K\boldsymbol{d} + \frac{1}{2}\boldsymbol{d}_x \cdot K_x\boldsymbol{d}_x + \boldsymbol{d} \cdot A\boldsymbol{d}_x, \tag{54}$$

where M is the mass parameter, G is the additional gyroscopic term

$$G = \frac{1}{2}\left(\dot{\boldsymbol{d}} \cdot B\boldsymbol{d} - \boldsymbol{d} \cdot B\dot{\boldsymbol{d}}\right), \tag{55}$$

$$\tag{56}$$

where $\boldsymbol{d}$ and $\boldsymbol{d}_x$ consist of internal variables and their gradients, respectively,

$$\boldsymbol{d} = \begin{bmatrix} u \\ \varphi \\ \psi \end{bmatrix} \quad \text{and} \quad \boldsymbol{d}_x = \begin{bmatrix} u_x \\ \varphi_x \\ \psi_x \end{bmatrix}, \tag{57}$$

and where B, K, K_x, and A are material parameters,

$$K = \begin{bmatrix} K^{uu} & K^{u\varphi} & K^{u\psi} \\ K^{u\varphi} & K^{\varphi\varphi} & K^{\varphi\psi} \\ K^{u\psi} & K^{\varphi\psi} & K^{\psi\psi} \end{bmatrix}, \quad K_x = \begin{bmatrix} K_x^{uu} & K_x^{u\varphi} & K_x^{u\psi} \\ K_x^{u\varphi} & K_x^{\varphi\varphi} & K_x^{\varphi\psi} \\ K_x^{u\psi} & K_x^{\varphi\psi} & K_x^{\psi\psi} \end{bmatrix}, \quad A = \begin{bmatrix} A^{uu} & A^{u\varphi} & A^{u\psi} \\ A^{\varphi u} & A^{\varphi\varphi} & A^{\varphi\psi} \\ A^{\psi u} & A^{\psi\varphi} & A^{\psi\psi} \end{bmatrix}. \tag{58}$$

The previous assumption needs some clarifying comments:

1) The choice of a Lagrangian is the first step in the D'Alembertian postulation. Very often the supporters of balance postulation ask why a specific Lagrangian is "true". To these supporters one should ask why balance laws are true and why constitutive equations are true and why Clausius-Duhem inequality must be true. We believe that the two approaches can be always treated in such a way that they are equivalent. Indeed, it is very well known that every scientific theory must be based on postulates which are basic assumptions whose validity can be proven only *a posteriori*. It is worth noticing that this is a particular case of an epistemological phenomenon that has been investigated by Pierre Duhem himself: that of underdetermination of a scientific theory (see [26]). A class of phenomena, indeed, does not uniquely determine the theoretical model which is able to describe it; on the contrary, the same phenomena can be "saved" by different models, and asking which one of these is "true" is often times pointless.

2) We will prove that the Euler-Lagrange equations deduced from the previous Lagrangian density functions are a generalization of the evolution equations found in the previous sections. We leave to anybody to decide which approach is preferable.

3) In particular, we underline that the role of the gyroscopic terms is very well known in Lagrangian mechanics. We found particularly interesting the treatment found in [39,5,12]. When one has a pair of Lagrangian parameters, whichever is their physical meaning, in order to have an exchange of energy between these two degrees of freedom, without dissipation, it is necessary to introduce a term in the kinetic energy as those we postulated in the expressions in G. Even if in the Euler-Lagrange equations one will find a term depending on first derivatives of Lagrange parameters, so that one of them may appear to be non-conservative, all together they will be conservative: Hamiltonian is constant along their solutions. Physically speaking, the energy disappearing from one degree of freedom reappears in another degree of freedom without loss. Many examples have been found in various applications exploiting gyroscopic coupling [1,56,37]: the name of the coupling being simply related to the first occurrence in the equations of the heavy top. Here, we want to explicitly cite the suggestive and careful book by Crandall [18].

The next step consists in writing the principle of minimum action as follows:

$$\delta \int_{t_0}^{t} \int_{-\infty}^{+\infty} \mathcal{L}\, dx\, dt = \delta \int_{t_0}^{t} \int_{-\infty}^{+\infty} (W - T)\, dx\, dt = 0, \tag{59}$$

where $\mathcal{L}$ is the Lagrangian density function, and we will assume a Hamiltonian structure for it, involving potential and kinetic energies. The first one will not depend on time derivatives, while the second one will be either quadratic or linear in these derivatives, in formulae

$$\mathcal{L}(\boldsymbol{d}, \dot{\boldsymbol{d}}, \boldsymbol{d}_x) = W(\boldsymbol{d}, \boldsymbol{d}_x) - T(\boldsymbol{d}, \dot{\boldsymbol{d}}) = W(\boldsymbol{d}, \boldsymbol{d}_x) - \left(T_1 \dot{\boldsymbol{d}} + \frac{1}{2} \dot{\boldsymbol{d}} \cdot T_2 \dot{\boldsymbol{d}} \right), \tag{60}$$

where T_1 and T_2 are linear functions of the Lagrangian coordinates $\boldsymbol{d}$.

The first variation of each term in the Lagrangian density function is obtained as follows:

$$\begin{aligned}
\int_{t_0}^{t}\int_{-\infty}^{+\infty} \delta W\, dx\, dt &= \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left(\frac{\partial W}{\partial \boldsymbol{d}}\delta \boldsymbol{d} + \frac{\partial W}{\partial \boldsymbol{d}_x}\delta \boldsymbol{d}_x\right) dx\, dt \\
&= \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left[\frac{\partial W}{\partial \boldsymbol{d}}\delta \boldsymbol{d} + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial \boldsymbol{d}_x}\delta \boldsymbol{d}\right) - \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial \boldsymbol{d}_x}\right)\delta \boldsymbol{d}\right] dx\, dt \\
&= \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left[\frac{\partial W}{\partial \boldsymbol{d}} - \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial \boldsymbol{d}_x}\right)\right] \delta \boldsymbol{d}\, \delta x\, \delta t, \\
\int_{t_0}^{t}\int_{-\infty}^{+\infty} \delta T\, dx\, dt &= \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left(\frac{\partial T}{\partial \boldsymbol{d}}\delta \boldsymbol{d} + \frac{\partial T}{\partial \dot{\boldsymbol{d}}}\delta \dot{\boldsymbol{d}}\right) dx\, dt \\
&= \int_{t_0}^{t}\int_{-\infty}^{+\infty} \left[\frac{\partial T}{\partial \boldsymbol{d}} - \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{d}}}\right)\right] \delta \boldsymbol{d}\, dx\, dt.
\end{aligned}$$

Then, by imposing that the condition given in Eqn. (59) is valid for every $\delta \boldsymbol{d}$, we get [1]

$$\frac{\partial W}{\partial \boldsymbol{d}} - \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial \boldsymbol{d}_x}\right) - \frac{\partial T}{\partial \boldsymbol{d}} + \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{d}}}\right) = 0, \tag{61}$$

where we used a compact vector form. Finally, by substituting the following previously calculated expressions

$$\begin{aligned}
\frac{\partial W}{\partial \boldsymbol{d}} &= K\boldsymbol{d} + A\boldsymbol{d}_x, \\
\frac{\partial W}{\partial \boldsymbol{d}_x} &= K_x\boldsymbol{d}_x + A^T\boldsymbol{d}, \\
\frac{\partial T}{\partial \boldsymbol{d}} &= -\frac{1}{2}\left(B_1 - B_1^T\right)\dot{\boldsymbol{d}}, \\
\frac{\partial T}{\partial \dot{\boldsymbol{d}}} &= M\dot{\boldsymbol{d}} + \frac{1}{2}\left(B_1 - B_1^T\right)\boldsymbol{d},
\end{aligned}$$

into Eqn. (61), the linear evolution equations (linearity being a consequence of the assumed quadratic form of Lagrangian density functions) are obtained as

$$K\boldsymbol{d} + (A - A^T)\boldsymbol{d}_x - K_x\boldsymbol{d}_{xx} + M\ddot{\boldsymbol{d}} + (B - B^T)\dot{\boldsymbol{d}} = 0. \tag{62}$$

Now, substuting the following particular choice of the matrices involved in the definition of Lagrangian,

$$M = \begin{bmatrix} \rho_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & B_2 \end{bmatrix}, \quad K_x = \begin{bmatrix} \rho_0 c^2 & A_1' & A_2' \\ A_1' & C_1 & 0 \\ A_2' & 0 & C_2 \end{bmatrix},$$

[1] We remark here that also in the mechanics of metamaterials the visionary statement found in Lagrange preface to analytical mechanics is true: "No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure. Those who appreciate mathematical analysis will see with pleasure mechanics becoming a new branch of it and hence, will recognize that I have enlarged its domain."

$$A = \begin{bmatrix} 0 & \frac{-A_1}{2} & \frac{-A_2}{2} \\ \frac{A_1}{2} & 0 & 0 \\ \frac{A_2}{2} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2R} \\ 0 & \frac{1}{2R} & 0 \end{bmatrix},$$

the final form of evolution equations, resulting to be equal to those obtained in the previous sections, is

$$\rho_0 \ddot{u} = \rho_0 c^2 u_{xx} + A_1' \varphi_{1_{xx}} + A_2' \varphi_{2_{xx}} + A_1 \varphi_{1_x} A_2 \varphi_{2_x}, \tag{63}$$

$$-\frac{\dot{\varphi}_2}{R} = C_1 \varphi_{1_{xx}} + A_1' u_{xx} - A_1 u_x - B_1 \varphi_1, \tag{64}$$

$$\frac{\dot{\varphi}_1}{R} = C_2 \varphi_{2_{xx}} + A_2' u_{xx} - A_2 u_x - B_2 \varphi_2. \tag{65}$$

It should be noted that this equivalence holds due to the absence of dissipation in the considered case.

7 Dispersion relation

In order to study dispersive effects, we derive the dispersion relations by assuming the solutions in the form of harmonic waves

$$u(x,t) = \hat{u} e^{i(kx-\omega t)}, \quad \varphi_1(x,t) = \hat{\varphi_1} e^{i(kx-\omega t)}, \quad \varphi_2(x,t) = \hat{\varphi_2} e^{i(kx-\omega t)}, \tag{66}$$

where k is the wavenumber, ω is the frequency, and $i^2 = -1$.

It should be noted that time derivatives of evolution equations

$$(\varphi_1)_{tt} = R(C_2(\varphi_2)_{txx} + A_2' u_{txx} - A_2 u_{tx} - B_2(\varphi_2)_t), \tag{67}$$

$$(\varphi_2)_{tt} = -R(C_1(\varphi_1)_{txx} + A_1' u_{txx} - A_1 u_{tx} - B_1(\varphi_1)_t), \tag{68}$$

can be rearranged by means of Eqs. (51) and (52) as follows

$$\begin{aligned}(\varphi_1)_{tt} = &- R^2 C_2 (C_1(\varphi_1)_{xx} + A_1' u_{xx} - A_1 u_x - B_1 \varphi_1)_{xx} + A_2' R u_{txx} - A_2 R u_{tx} + \\ &+ B_2 R^2 (C_1(\varphi_1)_{xx} + A_1' u_{xx} - A_1 u_x - B_1 \varphi_1),\end{aligned} \tag{69}$$

$$\begin{aligned}(\varphi_2)_{tt} = &- R^2 C_1 (C_2(\varphi_2)_{xx} + A_2' u_{xx} - A_2 u_x - B_2 \varphi_2)_{xx} - A_1' R u_{txx} + A_1 R u_{tx} + \\ &+ B_1 R^2 (C_2(\varphi_2)_{xx} + A_2' u_{xx} - A_2 u_x - B_2 \varphi_2).\end{aligned} \tag{70}$$

Substituting relations (66) into Eqs. (49), (69), and (70) we get

$$-\rho_0 \hat{u} \omega^2 = -\rho_0 c^2 \hat{u} k^2 + A_1 \hat{\varphi_1}(ik) - A_1' \hat{\varphi_1} k^2 + A_2 \hat{\varphi_2}(ik) - A_2' \hat{\varphi_2} k^2, \tag{71}$$

$$\begin{aligned}-\hat{\varphi_1}\omega^2 = &- R^2 C_2 [-C_1 \hat{\varphi_1} k^2 - A_1' \hat{u} k^2 - A_1 \hat{u}(ik) - B_1 \hat{\varphi_1}](-k^2) + A_2' R \hat{u}(i\omega) k^2 - \\ &- A_2 R \hat{u} \omega k + B_2 R^2 [-C_1 \hat{\varphi_1} k^2 - A_1' \hat{u} k^2 - A_1 \hat{u}(ik) - B_1 \hat{\varphi_1}],\end{aligned} \tag{72}$$

$$\begin{aligned}-\hat{\varphi_2}\omega^2 = &- R^2 C_1 [-C_2 \hat{\varphi_2} k^2 - A_2' \hat{u} k^2 - A_2 \hat{u}(ik) - B_2 \varphi_2](-k^2) - A_1' R \hat{u}(i\omega k^2) + \\ &+ A_1 R \hat{u} \omega k + B_1 R^2 [-C_2 \hat{\varphi_2} k^2 - A_2' \hat{u} k^2 - A_2 \hat{u}(ik) - B_2 \hat{\varphi_2}].\end{aligned} \tag{73}$$

The same can be represented in the matrix form

$$\begin{pmatrix} \rho_0 c^2 k^2 - \rho_0 \omega^2 & -iA_1 k + A_1' k^2 & -iA_2 k + A_2' k^2 \\ G_1 & F_1 & 0 \\ G_2 & 0 & F_2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{\varphi_1} \\ \hat{\varphi_2} \end{pmatrix} = 0, \tag{74}$$

where

$$G_1 = A_1' R^2 C_2 k^4 + A_1 R^2 C_2 (ik^3) - A_2' R (i\omega) k^2 + A_2 R \omega k + A_1' B_2 R^2 k^2 + A_1 B_2 R^2 (ik),$$

$$F_1 = R^2 C_2 C_1 k^4 + B_1 R^2 C_2 k^2 + B_2 R^2 C_1 k^2 + B_1 B_2 R^2 - \omega^2,$$

$$G_2 = A_2' R^2 C_1 k^4 + A_2 R^2 C_1 (ik^3) - A_1' R (i\omega) k^2 + A_1 R \omega k + A_2' B_1 R^2 k^2 + A_2 B_1 R^2 (ik),$$

$$F_2 = R^2 C_1 C_2 k^4 + B_2 R^2 C_1 k^2 + B_1 R^2 C_2 k^2 + B_2 B_1 R^2 - \omega^2,$$

In order to get nontrivial solutions the determinant of this system must vanish, i.e.,

$$(\rho_0 c^2 k^2 - \rho_0 \omega^2) F_1 F_2 + (iA_1 k - A_1' k^2) G_1 F_2 + (iA_2 k - A_2' k^2) F_1 G_2 = 0. \tag{75}$$

Since $F_1 = F_2$, the latter equation is slightly simplified

$$(\rho_0 c^2 k^2 - \rho_0 \omega^2) F_1 + (iA_1 k - A_1' k^2) G_1 + (iA_2 k - A_2' k^2) G_2 = 0, \tag{76}$$

and its explicit form can be reduced to

$$\begin{aligned} &\rho_0 c^2 k^2 (RC_2 C_1 k^4 + B_1 R C_2 k^2 + B_2 R C_1 k^2 + B_1 B_2 R - \omega^2/R) \\ &- \rho_0 \omega^2 (RC_2 C_1 k^4 + B_1 R C_2 k^2 + B_2 R C_1 k^2 + B_1 B_2 R - \omega^2/R) \\ &- A_1 A_1 R C_2 k^4 + A_1 A_2' \omega k^3 + i A_1 A_2 \omega k^2 - A_1 B_2 R A_1 k^2 \\ &- A_1' A_1' R C_2 k^6 + i A_1' A_2' \omega k^4 - A_1' A_2 \omega k^3 - A_1' A_1' B_2 R k^4 \\ &- A_2 A_2 R C_1 k^4 + A_1' A_2 \omega k^3 + i A_2 A_1 \omega k^2 - A_2 A_2 B_1 R k^2 \\ &- A_2' A_2' R C_1 k^6 + i A_2' A_1' \omega k^4 - A_2' A_1 \omega k^3 - A_2' A_2' B_1 R k^4 = 0. \end{aligned}$$

Real and imaginary parts should be equal to zero separately, which demands for the imaginary part

$$A_1 A_2 \omega k^2 + A_1' A_2' \omega k^4 + A_2 A_1 \omega k^2 + A_2' A_1' \omega k^4 = 0 \tag{77}$$

or

$$A_1 A_2 + A_1' A_2' k^2 = 0 \tag{78}$$

The obtained condition can be satisfied for arbitrary k in following cases:

1. $A_1 = A_1' = 0$;
2. $A_2 = A_2' = 0$;
3. $A_1 = A_2' = 0$;
4. $A_2 = A_1' = 0$;

Due to the symmetry of the adopted free energy with respect to internal variables, it is sufficient to consider only cases 2 and 3.

In the case 2, dispersion relation (76) is reduced to

$$\begin{aligned}&\rho_0 c^2 k^2 (C_2 C_1 k^4 + B_1 C_2 k^2 + B_2 C_1 k^2 + B_1 B_2 - \omega^2/R^2)\\&- \rho_0 \omega^2 (C_2 C_1 k^4 + B_1 C_2 k^2 + B_2 C_1 k^2 + B_1 B_2 - \omega^2/R^2)\\&- A_1 A_1 C_2 k^4 - A_1 A_1 B_2 k^2 - A'_1 A'_1 C_2 k^6 - A'_1 A'_1 B_2 k^4 = 0,\end{aligned}$$

while in the case 3 we have

$$\begin{aligned}&\rho_0 c^2 k^2 (C_2 C_1 k^4 + B_1 C_2 k^2 + B_2 C_1 k^2 + B_1 B_2 - \omega^2/R^2)\\&- \rho_0 \omega^2 (C_2 C_1 k^4 + B_1 C_2 k^2 + B_2 C_1 k^2 + B_1 B_2 - \omega^2/R^2)\\&- A'_1 A'_1 C_2 k^6 - A'_1 A'_1 B_2 k^4 - A_2 A_2 C_1 k^4 - A_2 A_2 B_1 k^2 = 0.\end{aligned}$$

Introducing a characteristic frequency ω_0 and a characterisric wavenumber k_0, we can represent the dispersion relation in the non-dimensional form

$$\begin{aligned}&(\rho_0 c^2 k_0^2 k^2 - \rho_0 \omega_0^2 \omega^2)(C_2 C_1 k_0^4 k^4 + B_1 C_2 k_0^2 k^2 + B_2 C_1 k_0^2 k^2 + B_1 B_2 - \omega_0^2 \omega^2/R^2)\\&- A_1 A_1 C_2 k_0^4 k^4 - A_1 A_1 B_2 k_0^2 k^2 - A'_1 A'_1 C_2 k_0^6 k^6 - A'_1 A'_1 B_2 k_0^4 k^4 = 0\end{aligned}$$

The natural choice for the connection between the characteristic frequency and the characteristic wavenumber is $\omega_0 = ck_0 = R$. For the convenience, we denote $C_2 k_0^2 = c_2, \quad C_1 k_0^2 = c_1$, which leads to

$$\begin{aligned}&(k^2 - \omega^2)(c_2 c_1 k^4 + B_1 c_2 k^2 + B_2 c_1 k^2 + B_1 B_2 - \omega^2)\\&- \frac{A_1 A_1 c_2 k_0^2}{\rho_0 c^2} k^4 - \frac{A_1 A_1 B_2 k_0^2}{\rho_0 c^2} k^2 - \frac{A'_1 A'_1 c_2 k_0^4}{\rho_0 c^2} k^6 - \frac{A'_1 A'_1 B_2 k_0^4}{\rho_0 c^2} k^4 = 0.\end{aligned}$$

Now we introduce

$$\frac{A_1 A_1 k_0^2}{\rho_0 c^2} = a_1^2, \quad \frac{A'_1 A'_1 k_0^4}{\rho_0 c^2} = {a'}_1^2$$

and have finally a quadratic equation for $k^2 - \omega^2$

$$\begin{aligned}&(k^2 - \omega^2)^2 + (k^2 - \omega^2)(c_2 c_1 k^4 + B_1 c_2 k^2 + B_2 c_1 k^2 + B_1 B_2 - k^2)\\&- a_1^1 c_2 k^4 - a_1^1 B_2 k^2 - {a'}_1^1 c_2 k^6 - {a'}_1^1 B_2 k^4 = 0,\end{aligned} \tag{79}$$

which has the standard solution

$$\begin{aligned}&(\omega^2 - k^2) = 0.5(c_2 c_1 k^4 + B_1 c_2 k^2 + B_2 c_1 k^2 + B_1 B_2 - k^2)\\&\pm \sqrt{0.25(c_2 c_1 k^4 + B_1 c_2 k^2 + B_2 c_1 k^2 + B_1 B_2 - k^2)^2 + a_1^2 c_2 k^4 + a_1^2 B_2 k^2 + {a'}_1^2 c_2 k^6 + {a'}_1^2 B_2 k^4}.\end{aligned} \tag{80}$$

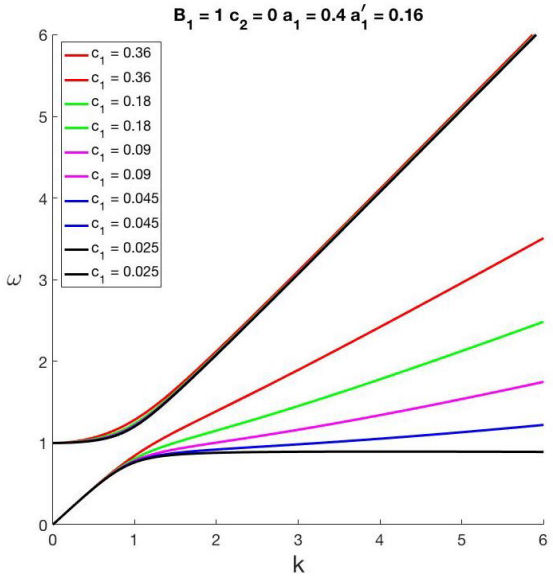


Fig. 1: Variation of dispersion curves depending on values of the dimensionless parameter c_1

8 Band gaps in microstructured materials

In this section, we investigate the effects of different parameters on dispersion curves and band gaps in microstructure. As can be seen from Eq. (80) there are 6 independent material parameters, and in order to analyse their roles, we start with $c_2 = 0$, which coincides with the unified microstructure model [8]

$$\begin{aligned}(\omega^2 - k^2) &= 0.5(B_2c_1k^2 + B_1B_2 - k^2)\\ &\pm \sqrt{0.25(B_2c_1k^2 + B_1B_2 - k^2)^2 + a_1^2B_2k^2 + a'^2_1B_2k^4}.\end{aligned} \tag{81}$$

We choose the starting point of the optical branch in such a way that it corresponds to a unit frequency. Therefore, we have $B_1B_2 = 1$, as it follows from Eq. (81) and we begin with investigating the effects of c_1 on the dispersion curves by choosing the following parameters: $B_1 = 1$, $c_2 = 0$, $a_1 = 0.4$, and $a_1' = 0.16$. As can be seen from Fig. 1, while the optical branch, in the considered case, changes slightly with different c_1 values, the acoustic branch varies significantly, and a band gap is observed with the lower values of c_1. In Figs. 2 and 3, we present the effects of the dimensionless parameters a_1 and a_1' on the band gap obtained with $c_1 = 0.025$ in the previous considered case. It is clearly seen that the band gap expands when the value of a_1 increases. However, the band gap does not change significantly with different a_1' values, and here, we again observed that the optical branch is not considerably affected by different values of a_1 and a_1'.

The influence of dimensionless parameters c_2 is studied in Fig. 4. It is clear that the band gap disappears with increasing c_2 values, and the optical branch varies slightly. We also studied the effects of the dimensionless parameter B_1 on

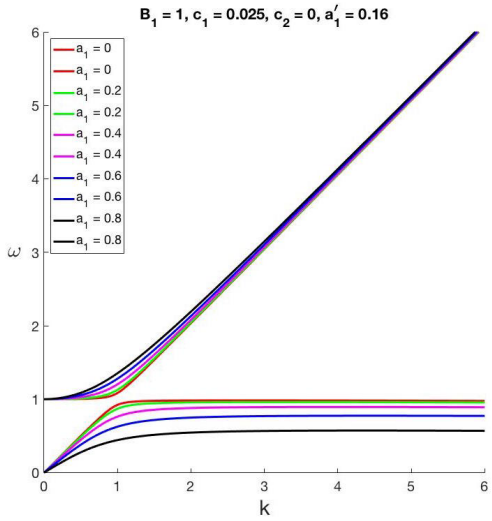


Fig. 2: Variation of dispersion curves depending on values of the dimensionless parameter a_1

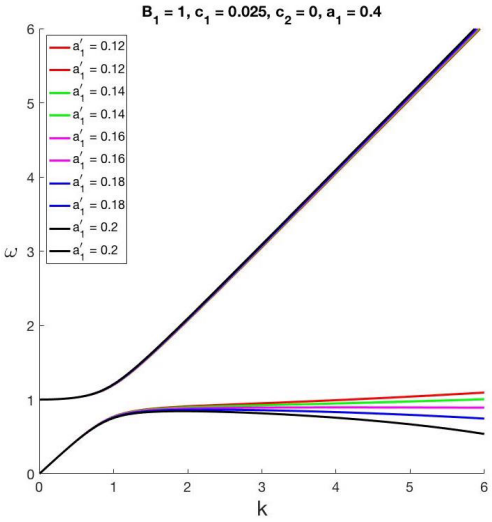


Fig. 3: Variation of dispersion curves depending on values of the dimensionless parameter a_1'

the band gap. In Fig. 5, we observe that the lower values of B_1, especially the values less than 1, may affect not only the band gap but also the optical branch, the widest band gap is obtaned with $B_1 = 0.2$. On the other hand, higher values of B_1 (the values greater than 1) do not significantly affect the dispersion curves, and so the band gap.

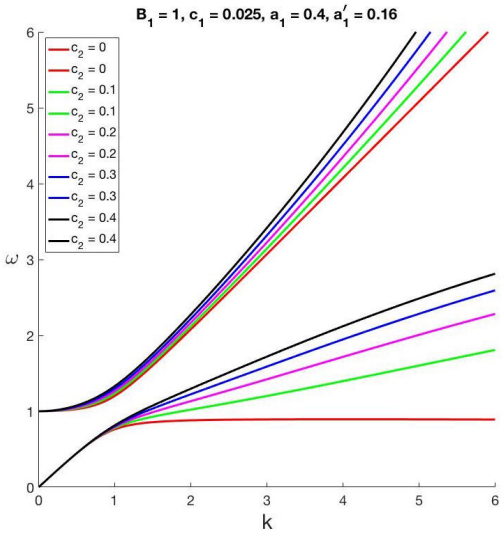


Fig. 4: Variation of dispersion curves depending on values of the dimensionless parameter c_2

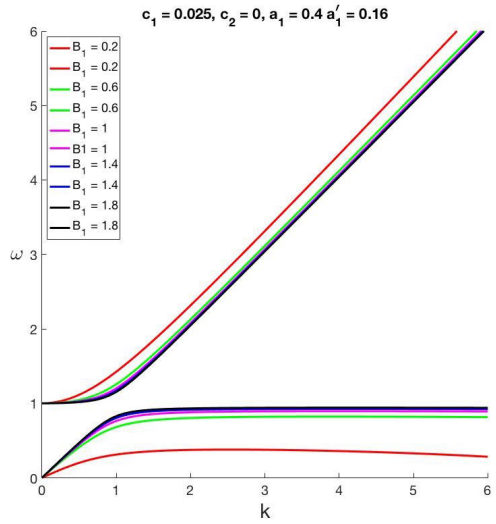


Fig. 5: Variation of dispersion curves depending on values of the dimensionless paramater B_1

We also elaborated an efficient code to systematically look for band gaps in the plots of frequency dependence on wave numbers in planar waves. In our code, we consider full dispersion relation Eq. (80) with $B_2 = \frac{1}{B_1}$. This is a novel situation with respect to the ones considered previously in the literature. Indeed, the existence of two internal variables (φ_1, φ_2) implies the existence of two possible internal elastic stiffnesses (B_1, B_2), which because of gyroscopic effect influence

the derivative of the other internal degree of freedom. Now, we want to find which band gaps exist in the dispersion relationships within some determined ranges for the parameters B_1, c_1, c_2, a_1, and a_1'. In order to do this we conceived a computational procedure which adopts the following flowchart

Algorithm 1 Band Gaps

1. Choose the ranges for B_1, c_1, c_2, a_1, a_1' to be examined
2. Calculate the corresponding range of B_2
3. Choose a tolerance, $TOL > 0$
4. Start a for loop over parameter ranges
   - Calculate ω^+ (optical branch) and ω^- (acoustic branch)
   - if $\omega^+ - \omega^- > TOL$ and $\text{imag}(\omega^+) = 0$ and $\text{imag}(\omega^-) = 0$
   - then store the corresponding B_1, B_2, c_1, c_2, a_1, a_1' values, and the thickness of the band gap.
   - End the for loop
5. Find the maximum band gap thickness

We remark that we did not explicitly state our condition for the existence of a energy conservation principle for the family of Lagrangian density functions introduced in Eq. (60). However, we have imposed that the plane waves propagate in the considered material without being neither damped nor forced so that for a given wave number the frequency must be assumed to be real.

The influence of dimensionless parameters B_1 and c_2 is studied in Figs. 4 and 5. It is clear that the band gap contracts slightly with increasing c_2 values while it is expanded with lower values of B_1.

We conclude this section by observing that the described algorithm allows us to determine some values of the constitutive parameters for which the band gap has as amplitude the value 0.8622 which seems to us to be close to the optimal one. More investigation about this point is postponed to further investigation. The value of the parameters calculated and the shape of the dispersion formulae are shown in the following figure.

9 Propagation of wave trains in conceived metametarial

In this section, the behaviour of the internal variables is investigated by solving the partial differential equations given by Eqs. (63), (64), and (65) with COMSOL Multiphysics® considering a one-dimensional domain with a length of 20 m, density $\rho_0 = 1$, and the coefficients determined by the condition of maximality of band gap amplitude. The simulation has been performed with the standard package Weak Form PDE and therefore we do not need to describe it. We have imposed for a very long slab of metamaterial on the left side boundary conditions fixing equal to zero the value of all fields and we left free their derivatives. On the other hand, on the right boundary an imposed displacement is assigned as a function of time as specified in Fig. 7,

$$u(t) = A_0 \,\text{sech}\left(\kappa\left[-\alpha T_s + t\right]\right) \tanh\left(\kappa\left[-\alpha T_s + t\right]\right) + R_0 \tag{82}$$

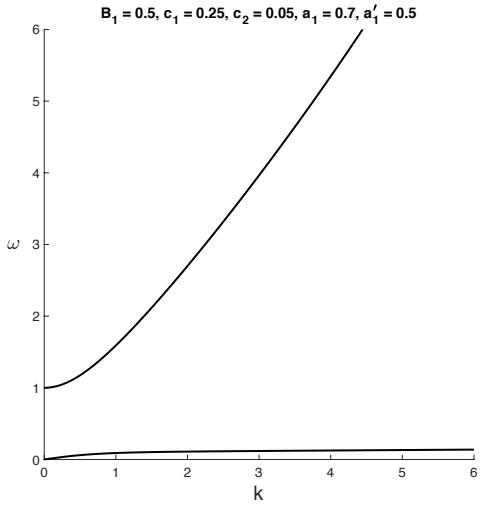


Fig. 6: Optimal band gap

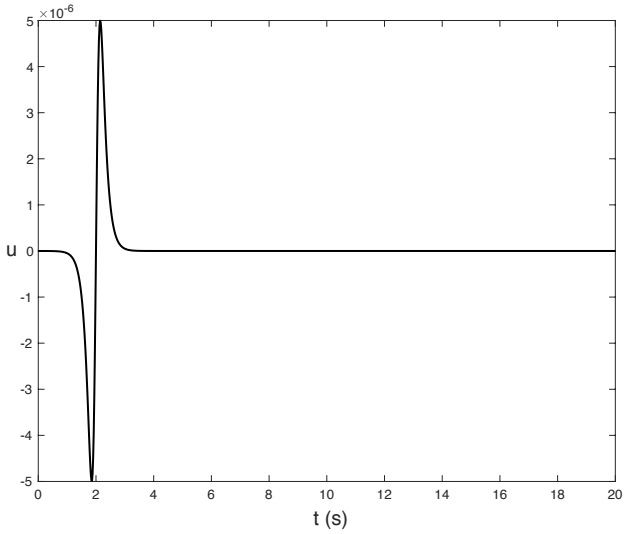
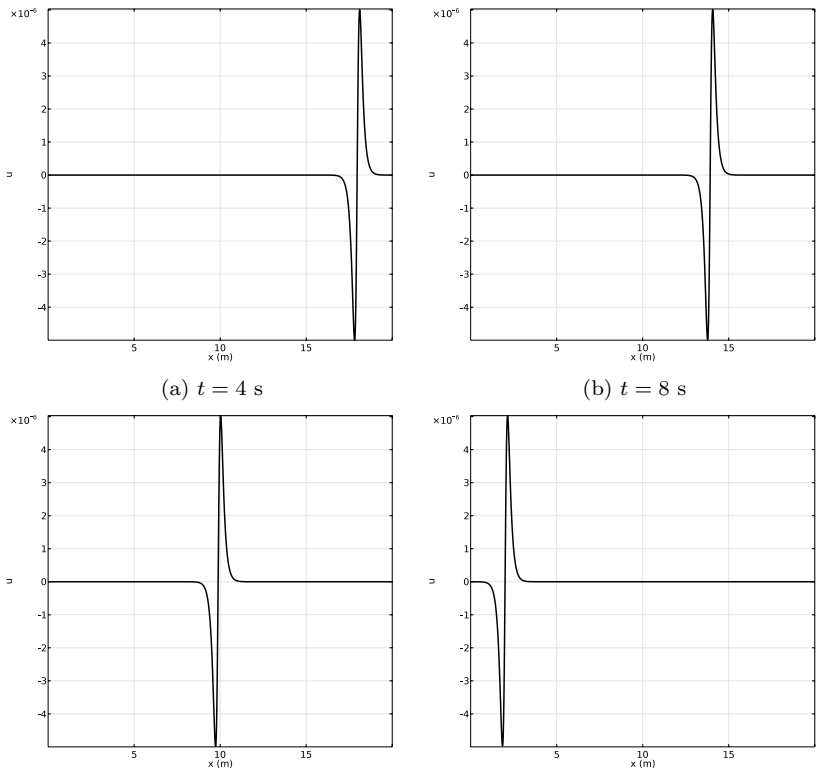


Fig. 7: Time-varying displacement boundary condition on the right side of the slab

where $t \in [0, T_s]$, and T_s is the total simulation time which is 20 s in this study, α is 0.1, κ is 6 s^{-1}, A_0 is 10^{-5} m and R_0 is a constant to impose that $u(0)$ vanishes. The internal variables are again assigned to be zero at the right side, with all derivatives being left arbitrary.

Finally, we have chosen as constitutive parameters those for which the band gap is the widest possible ($B_1 = 0.5$, $c_1 = 0.25$, $c_2 = 0.05$, $a_1 = 0.7$, $a_1' = 0.5$). We are aware that a complete Fourier analysis is needed of the incoming signal to describe how the band gap cuts some of the frequencies contained in the Fourier spectrum



(a) $t = 4$ s (b) $t = 8$ s

(c) $t = 12$ s (d) $t = 20$ s

Fig. 8: Displacement progress without internal variables

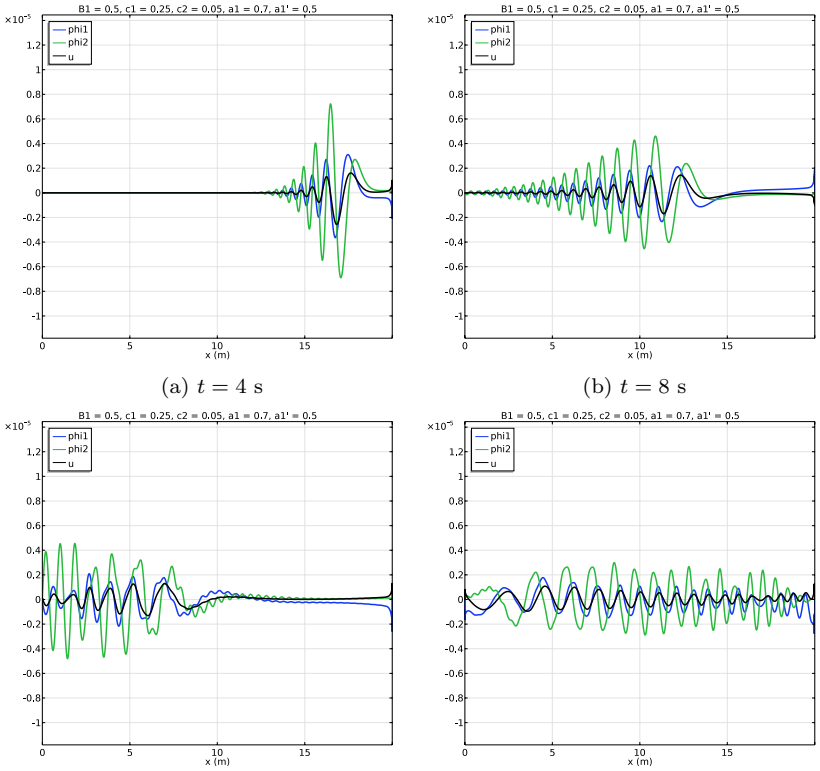
of incoming signal, and we are also aware that standing evanescent waves may arise inside the slab varying periodically with time: we also postpone this analysis to later investigations where we will more carefully consider the meaning of the equations discussed after Eq. (78). In Fig. 8, we show how the displacement signal progresses inside the slab when it is not coupled to the internal variables ($R = 0$).

This simulation will give us a benchmark for estimating the attenuation capacity of the designed metametarial. Indeed, with the optimal parameters already found, in Fig. 9, we show how, in the considered slab, the excitation of the vibrations of internal parameters is able to attenuate the amplitude and slow down the progressing speed of the displacement wave.

Finally, in Fig. 10, the total energy is plotted as a function of time to verify that there is no dissipation and so the system is conservative as it is expected.

10 Conclusion

The results presented in this paper are both theoretical and numerical. Indeed, we proved that Lagrangian variational principles and Clausius-Duhem thermodynamics give consistent predictions and exactly the same evolution equations also for



(a) $t = 4$ s (b) $t = 8$ s

(c) $t = 12$ s (d) $t = 20$ s

Fig. 9: Displacement progress with internal variables

microstructured metametarials with two internal variables. On the other hand, we have shown how one can optimize the constitutive parameters of a precise class of metametarials to get a numerically optimized band gap for plane non-evanescent waves. It is a remarkable observation the one which we obtained at the end of the previous section: a slab of optimized metametarial, even in the absence of damping and dissipation, can slow down a progressing displacement wave. Many questions were left open for later investigations which did not appear of relevance before. We list few of them: 1) It has to be cleared up the role of evanescent standing waves and their relation with the conditions after Eqn (78), 2) boundary conditions for finite slab and their influence in attenuation of displacement waves must be carefully considered, 3) non-linearities and damping must be included in the modeling, 4) inertia for internal variables must be generalized with a mass matrix M which is positive definite and its influence on wave phenomena has to be investigated, 5) the evolution equations deduced from the most complete Lagrangian, including all gyroscopic effects, have to be studied and optimized, probably showing even more effective band gaps.

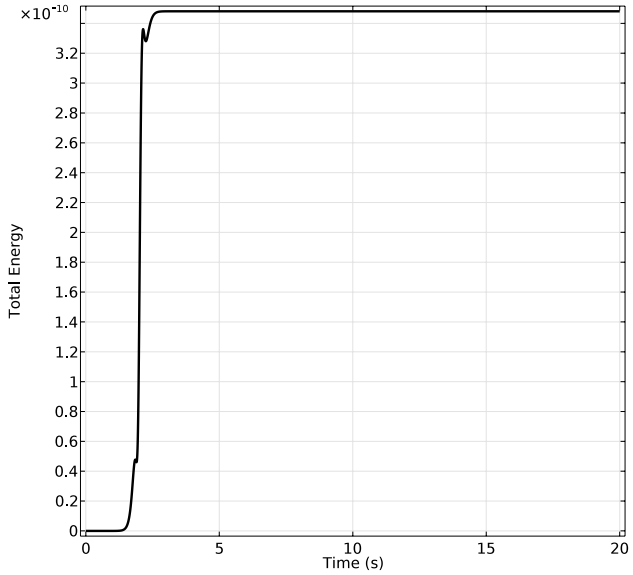


Fig. 10: Total energy as a function of time

Acknowledgements AB was supported by the EU through the European Regional Development Fund and by the Estonian Research Council grant PUT434. DS is supported by a grant from the Government of the Russian Federation (contract No. 14.Y26.31.0031).

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