Causality in strain gradient elasticity: an internal variables approach

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Abstract

The absence of higher order time derivatives in equations of motion causes the infinite velocity of wave propagation in strain gradient elasticity models. This issue needs to be avoided from both a theoretical and practical viewpoint. It is shown that dual internal variables approach solves the causality problem. Furthermore, the primary internal variable can be interpreted as the Laplacian of the strain gradient resulting in an Aifantis-type strain gradient model.

Keywords: Causality, strain gradient elasticity, internal variables

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1. Introduction

The strain gradient elasticity theory can be considered as a low frequency, long wavelength approximation of the micromorphic elasticity [1] with vanishing relative deformation [2, 3]. The corresponding free energy density W for linear isotropic elasticity can be represented as [2]

$$\begin{aligned} W =& \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + a_1\varepsilon_{ik,i}\varepsilon_{jj,k} + a_2\varepsilon_{jj,i}\varepsilon_{kk,i} + \\ &+ a_3\varepsilon_{ik,i}\varepsilon_{jk,j} + a_4\varepsilon_{jk,i}\varepsilon_{jk,i} + a_5\varepsilon_{jk,i}\varepsilon_{ij,k}, \end{aligned} \tag{1}$$

where ε_{ij} is the strain tensor, λ and μ are Lamé constants, a_i are additional constitutive parameters, the notation $(\cdot)_{,i}$ denotes derivative.

Limitations of such an approximation are related to the absence of additional internal degrees of freedom [3]. For instance, only acoustic branch of the corresponding dispersion curve can be revealed in the dynamical case of the strain gradient theory [4].

Another approach to the strain gradient elasticity is based on the theory of nonlocal elasticity in which the integrals are replaced by gradients [5]. It is referred to as a Laplacian-based theory of gradient elasticity [6] because the

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Laplacian of the stress is added to the linear elastic constitutive equations [7]. Aifantis and coworkers suggested to include in the constitutive equations the Laplacian of the strain instead [8, 9].

This idea has been elaborated extensively by Askes and coworkers [6, 10–12] as well as by other researchers [13, 14]. Though this theory is able to avoid singularities in strain components at a sharp crack tip, the optical branch of the dispersion curve is still lost. The reason of that is the absence of higher order time derivatives in the equations of motion. This leads to the causality problem [15], which involves the possibility of infinite wave propagation speed. The solution of the causality problem is to either add the higher order time derivatives phenomenologically [12, 15] or obtain them using internal variables [16–18] but without regard for the strain gradients.

The purpose of this paper is to find out a possible relationship between internal variables and strain gradients in order to avoid the causality problem. It is required due to the use strain gradients in numerous applications [6, 19].

As demonstrated in Section 2 for the one-dimensional case, the introduction of a single internal variable is insufficient to eliminate the causality issue. The dual internal variables approach can be used to rule out this problem. However, the one-dimensional case does not directly lead to the interpretation of internal variables in terms of strain gradients. Such an interpretation is only possible in three dimensions (Section 3).

2. One-dimensional case

We begin with the one-dimensional case due to its simplicity and wide usability [10]. The one-dimensional linear elastic motion of homogeneous solid heat conductors is governed by the balance of linear momentum (no body forces)

$$\rho v_{,t} - \sigma_{,x} = 0, \tag{2}$$

and by the energy balance

$$\left(\frac{1}{2}\rho v^2 + \mathcal{E}\right)_{,t} - (\sigma v - Q)_{,x} = 0. \tag{3}$$

The balance equations are complemented by the second law of thermodynamics

$$S_{,t} + \left(\frac{Q}{\theta}\right)_{,x} \geq 0. \tag{4}$$

Here ρ is the matter density, v is the particle velocity, σ is the one-dimensional stress, $\mathcal{E}$ is the internal energy density, Q is the heat flux, S is the entropy per unit volume, θ is temperature.

In terms of the free energy density $W = \mathcal{E} - S\theta$, the second law (4) can be represented as

$$-W_{,t} + v_{,x}\sigma - S\theta_{,t} - \frac{Q}{\theta}\theta_{,x} \geq 0. \tag{5}$$

We must now specify the free energy density using constitutive assumptions. Classically, the free energy density depends on the strain and temperature $W = \overline{W}(\varepsilon, \theta)$, which, together with definitions of the stress and entropy

$$\sigma = \frac{\partial \overline{W}}{\partial \varepsilon}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \tag{6}$$

reduces dissipation inequality (5) to $-Q\theta_{,x}/\theta \geq 0$, which is the basis for the Fourier law. The constitutive assumption $W = \overline{W}(\varepsilon, \theta)$, is valid for homogeneous solids. To account for the influence of possible inhomogeneity, the state space will be extended by internal variables of state.

2.1. Single internal variable

The standard method for generalizing the continuous description is to introduce new variables into the state space. In the simplest case, the state space is extended by an internal variable α whose physical meaning is left unspecified for the time being

$$W = \overline{W}(\varepsilon, \theta, \alpha). \tag{7}$$

Then the time derivative of the free energy density

$$W_{,t} = \frac{\partial \overline{W}}{\partial \theta}\theta_{,t} + \frac{\partial \overline{W}}{\partial \varepsilon}\varepsilon_{,t} + \frac{\partial \overline{W}}{\partial \alpha}\alpha_{,t}, \tag{8}$$

transforms the dissipation inequality to the form

$$-\frac{\partial \overline{W}}{\partial \alpha}\alpha_{,t} - \frac{Q}{\theta}\theta_{,x} \geq 0. \tag{9}$$

If the free energy is a quadratic function of state variables (in isothermal case)

$$\overline{W} = \frac{\rho c^2}{2}\varepsilon^2 + A\alpha\varepsilon + \frac{1}{2}B\alpha^2, \tag{10}$$

then the dissipation inequality is reduced to

$$-(A\varepsilon + B\alpha)\alpha_{,t} \geq 0. \tag{11}$$

The solution of dissipation inequality (11) represents the evolution equation of the internal variable

$$\alpha_{,t} = -k(A\varepsilon + B\alpha), \quad k \geq 0. \tag{12}$$

The non-dissipative case corresponds to the relationship $\alpha = -A\varepsilon/B$. Because the stress in this case is

$$\sigma = \frac{\partial \overline{W}}{\partial \varepsilon} = \rho c^2 \varepsilon + A\alpha, \tag{13}$$

equation of motion (2) in terms of displacement u

$$\rho u_{,tt} = \rho c^2 u_{,xx} - \frac{A^2}{B}u_{,xx}, \tag{14}$$

is the wave equation with the slowing down velocity of propagation. As a result, extending the state space by a single internal variable yields a non-dispersive wave equation with no relationship to strain gradients. Furthermore, the weakly nonlocal version of the single internal variable theory analysed in [17] results in dispersive wave equations similar to those in the strain gradient theory discussed in [6].

2.2. Dual internal variables

The inadequacy of the single internal variable approach to describing dispersive waves forces us to go further. The next step is to employ the dual internal variable treatment [20]. We apply the simplest variant of the dual internal variables theory in which the state space is enlarged by internal variables α and β

$$W = \overline{W}(\varepsilon, \theta, \alpha, \beta). \tag{15}$$

More information on the dual internal variables approach can be found in [21].

Calculating the time derivative of the free energy density using chain rule

$$W_{,t} = \frac{\partial \overline{W}}{\partial \theta}\theta_{,t} + \frac{\partial \overline{W}}{\partial \varepsilon}\varepsilon_{,t} + \frac{\partial \overline{W}}{\partial \alpha}\alpha_{,t} + \frac{\partial \overline{W}}{\partial \beta}\beta_{,t}, \tag{16}$$

and substituting the result into inequality (5), we obtain the dissipation inequality in the form

$$-\frac{\partial \overline{W}}{\partial \alpha}\alpha_{,t} - \frac{\partial \overline{W}}{\partial \beta}\beta_{,t} - \frac{Q}{\theta}\theta_{,x} \geq 0. \tag{17}$$

In the isothermal case, the inequality includes only terms that depend on internal variables

$$-\frac{\partial \overline{W}}{\partial \alpha}\alpha_{,t} - \frac{\partial \overline{W}}{\partial \beta}\beta_{,t} \geq 0. \tag{18}$$

To be more specific, we assume that the free energy is a quadratic function

$$\overline{W} = \frac{\rho c^2}{2}\varepsilon^2 + A\alpha\varepsilon + \frac{1}{2}B\alpha^2 + \frac{1}{2}C\beta^2. \tag{19}$$

Such a dependence transforms dissipation inequality (18) to

$$-(A\varepsilon + B\alpha)\alpha_{,t} - C\beta\beta_{,t} \geq 0. \tag{20}$$

It is easy to see that the choice

$$\alpha_{,t} = -MC\beta, \quad \beta_{,t} = M(A\varepsilon + B\alpha), \tag{21}$$

provides zero dissipation. The constant M is arbitrary and can be set to 1 without loss of generality. At this stage, the internal variable β can be eliminated by differentiation $(21)_1$ with respect to time and accounting for $(21)_2$

$$\alpha_{,tt} = -C(A\varepsilon + B\alpha). \tag{22}$$

Calculating the value of stresss

$$\sigma = \frac{\partial \overline{W}}{\partial \varepsilon} = \rho c^2 \varepsilon + A\alpha, \tag{23}$$

we represent the balance of linear momentum in terms of the displacement u as

$$\rho u_{,tt} = \rho c^2 u_{,xx} + A\alpha_{,x}. \tag{24}$$

Combining evolution equation of the internal variable α (22) and balance of linear momentum (24), we obtain the single equation of motion

$$\rho u_{,tttt} = \rho c^2 u_{,xxtt} - CA^2 u_{,xx} - CB(\rho u_{,tt} - \rho c^2 u_{,xx}). \tag{25}$$

It is clear that Eq. (25) is valid for dispersive waves propagation and has no problem with causality. To demonstrate this, we perform the dispersion analysis. First, we introduce dimensionless variables using a characteristic length L and a characteristic time T_0

$$X = \frac{x}{L}, \quad T = \frac{t}{T_0}, \quad U = \frac{u}{L}. \tag{26}$$

Then the choice of the governing parameters $T_0^2 = L^2/c^2$, $C = 1/T_0^2$, $A^2 = \rho c^2$, transforms equation of motion (25) to the dimensionless form

$$U_{,TTTT} = U_{,XXTT} - U_{,XX} - B(U_{,TT} - U_{,XX}). \tag{27}$$

Using harmonic waves with frequency ω and wavenumber k, $U(X,T) = \hat{U} e^{i(kX - \omega T)}$, we arrive at the dispersion relation

$$\omega^4 - \omega^2(k^2 + B) - (1 - B)k^2 = 0. \tag{28}$$

Dispersion curves in the non-dimensional case are presented in Fig. 1 for $B = 2$. There are both acoustic and optical branches present, with a bandgap between them. The connection to the strain gradient theory is not obvious, however. To clarify the situation, let us address to the three-dimensional formulation of the problem.

3. Three-dimensional case

The local balance laws in the small-strain approximation (no body forces) are

$$\frac{\partial(\rho v_i)}{\partial t} - \sigma_{ij,j} = 0, \tag{29}$$

$$\frac{\partial(K + \mathcal{E})}{\partial t} - (\sigma_{ij} v_{i,j} - Q_{i,i}) = 0, \tag{30}$$

and the second law of thermodynamics has the form

$$\frac{\partial S}{\partial t} + \left(\frac{Q_i}{\theta}\right)_{,i} \geq 0. \tag{31}$$

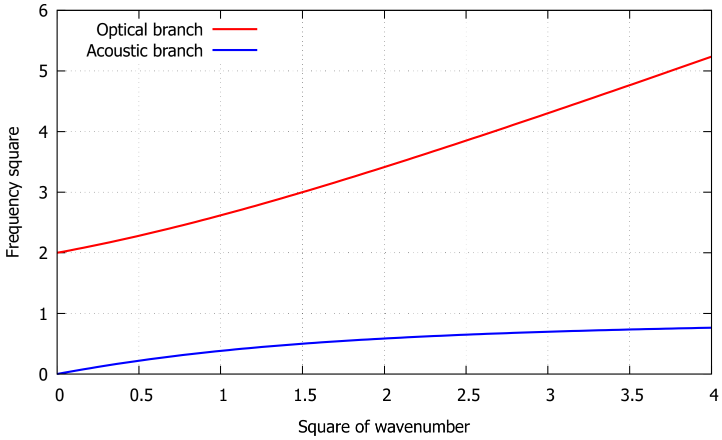

Figure 1: Dispersion curves.

Here $K = \rho {v_i}^2/2$ is the kinetic energy density and $\mathcal{E}$ is the internal energy per unit volume.

We apply the dual internal variable approach to avoid violation of causality. Typically, internal variables are introduced without specifying their tensorial nature. To be more specific, internal variables will be treated as second-order tensors. Then the free energy per unit volume W is represented as a sufficiently regular function

$$W = \overline{W}(\varepsilon_{ij}, \theta, \alpha_{ij}, \beta_{ij}). \tag{32}$$

The corresponding equations of state are given by

$$\sigma_{ij} = \frac{\partial \overline{W}}{\partial \varepsilon_{ij}}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad A_{ij} := -\frac{\partial \overline{W}}{\partial \alpha_{ij}}, \quad B_{ij} := -\frac{\partial \overline{W}}{\partial \beta_{ij}}. \tag{33}$$

The accepted functional dependence (32) and the equations of state (33) lead to the time derivative of the free energy in the form

$$\begin{aligned} \frac{\partial \overline{W}}{\partial t} =& \frac{\partial \overline{W}}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial t} + \frac{\partial \overline{W}}{\partial \theta} \frac{\partial \theta}{\partial t} + \frac{\partial \overline{W}}{\partial \alpha_{ij}} \frac{\partial \alpha_{ij}}{\partial t} + \frac{\partial \overline{W}}{\partial \beta_{ij}} \frac{\partial \beta_{ij}}{\partial t} = \\ =& \sigma_{ij} \dot{\varepsilon}_{ij} - S\dot{\theta} - A_{ij}\dot{\alpha}_{ij} - B_{ij}\dot{\beta}_{ij}. \end{aligned} \tag{34}$$

Here dot over a symbol denotes time derivative. The corresponding dissipation inequality is reduced in the isothermal case to

$$A_{ij}\dot{\alpha}_{ij} + B_{ij}\dot{\beta}_{ij} \geq 0. \tag{35}$$

If the evolution equations for the two internal variables are coupled by the relationships

$$\dot{\alpha}_{ij} = B_{ij}, \quad \dot{\beta}_{ij} = -A_{ij}, \tag{36}$$

then the dissipation disappears, just as it does in the scalar case. In this case, the evolution of one internal variable is driven by another one that indicates the duality between the internal variables. Representing the free energy as a quadratic function of its arguments

$$\overline{W} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + P\varepsilon_{ij}\alpha_{ij} + \frac{Q}{2}\alpha_{ij}\alpha_{ij} + \frac{R}{2}\beta_{ij}\beta_{ij}, \tag{37}$$

we arrive at corresponding equations of state

$$\begin{aligned} \sigma_{ij} &= \frac{\partial\overline{W}}{\partial\varepsilon_{ij}} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + P\alpha_{ij}, \\ A_{ij} &= -\frac{\partial\overline{W}}{\partial\alpha_{ij}} = -P\varepsilon_{ij} - Q\alpha_{ij}, \quad B_{ij} = -\frac{\partial\overline{W}}{\partial\beta_{ij}} = -R\beta_{ij}. \end{aligned} \tag{38}$$

The balance of linear momentum can be written in terms of the displacement

$$\rho\ddot{u}_i = (\lambda+\mu)u_{j,ij} + \mu u_{i,jj} + P\alpha_{ij,j}. \tag{39}$$

Elimination of the second internal variable using equations (36) determines the evolution equation for the internal variable α

$$\ddot{\alpha}_{ij} = -R\dot{\beta}_{ij} = RA_{ij} = -RP\varepsilon_{ij} - RQ\alpha_{ij}. \tag{40}$$

Furthermore, we can write down a single equation of motion in terms of the displacement

$$\rho\ddddot{u}_i - (\lambda+\mu)\ddot{u}_{j,ij} - \mu\ddot{u}_{i,jj} = -RP^2\frac{1}{2}(u_{i,jj}+u_{j,ij}) - RQ(\rho\ddot{u}_i - (\lambda+\mu)u_{j,ij} - \mu u_{i,jj}). \tag{41}$$

It remains to interpret the internal variable. If we suppose that the internal variable is a function of the strain gradient, i.e., $\alpha_{ij} = \alpha(\varepsilon_{ij,k})$, then the second rank tensor α_{ij} should be expressed in terms of the third rank tensor $\varepsilon_{ij,k}$. The natural choice of such a relationship is

$$\alpha_{ij} = k\varepsilon_{ij,kk} = k\Delta\varepsilon_{ij}, \tag{42}$$

k is an appropriate constant. Definition of the stress $(38)_1$ leads to the Aifantis strain gradient model [6, 8, 9]

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + kP\Delta\varepsilon_{ij}. \tag{43}$$

Furthermore, the causality problem is not present in equation of motion (41) as it is in the one-dimensional case.

4. Concluding remarks

Exploitation of dual internal variables eliminates the causality issues in the corresponding dispersive wave equation in the framework of strain gradient elasticity. Moreover, identifying the internal variable with the Laplacian of the strain leads to the Aifantis model for the strain gradient elasticity. The presented model does not replace existing strain gradient models, because it does not explicitly include the third-order strain gradient tensor. The model can be combined with available strain gradient models as a part containing only second-order tensors.

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