Inverse problems for parabolic integro-differential equations with instant and integral conditions

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Physical notation

Let us introduce the following parabolic integro-differential equation:

$$\beta[u + \mu * u]_t \;=\; Au - m * Au + \chi, \tag{1}$$

where χ is the source term, the symbol $*$ stands for the time convolution $v_1 * v_2(t) = \int_0^t v_1(t-\tau)v_2(\tau)d\tau$ and

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j} + a(x,t), \tag{2}$$

where a_{ij}, a_j and a are some coefficients.
We will consider the solution u of the integro-differential equation (1) for the arguments

$$(x,t) \in Q = \Omega \times (0,T),$$

where $\Omega \in \mathbb{R}^n$ is an n-dimensional open domain and $T > 0$ is a fixed number.

Smooth problems

We will start with the initial-boundary value problem

$$\beta[u + \mu * u]_t = Au - m * Au + \chi \quad \text{in } Q, \tag{3}$$

$$u = u_0 \text{ in } \Omega \times \{0\}, \quad Bu = b \text{ in } S = \Gamma \times (0, T), \tag{4}$$

where Γ is the boundary of Ω, u_0 and b are given functions, B is the boundary operator defined either by

$$Bu = u \qquad \text{(we call it } \textit{case } \text{I)} \tag{5}$$

or by

$$Bu = \omega \cdot \nabla u - m * \omega \cdot \nabla u \quad \text{(we call it } \textit{case } \text{II)}, \tag{6}$$

$\omega(x) = (\omega_1(x), \ldots, \omega_n(x))$ is an x-dependent vector satisfying the condition $\omega \cdot \nu > 0$ and $\nu(x)$ is the outer normal of Γ at the point $x \in \Gamma$. We assume that $\omega \in (C^1(\Gamma))^n$.

Throughout the talk we assume that the x-dependent coefficient matrix a_{ij} of the higher order part of the operator A is uniformly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \geq \epsilon|\lambda|^2 \quad \text{in } \Omega \ \text{ for any } \lambda \in \mathbb{R}^n \ \text{ and some } \epsilon \in (0,\infty) \qquad (7)$$

and x-dependent coefficient β is strictly positive:

$$\beta \geq \beta_0 \quad \text{in } \Omega \ \text{ with some } \beta_0 \in (0,\infty). \qquad (8)$$

In the analysis we will make use of the Hölder spaces $C^l(\Omega)$ and anisotropic Hölder spaces $C^{l,\frac{l}{2}}(Q)$. The norms in these spaces are denoted by $\|\cdot\|_l$ and $\|\cdot\|_{l,\frac{l}{2}}$, respectively.

Let us formulate the following inverse problems that use over-determined final data at $t = T$ of the solution of (3), (4):

IP1: Let the free term be of the following form:

$$\chi(x,t) = z(x)\phi(x,t) + \chi_0(x,t). \tag{9}$$

Given $\mu, m, \beta, a_{ij}, a_j, a, u_0, b, \phi, \chi_0$ and a function $u_T(x)$, $x \in \Omega$, **find** z **and** u so that the relations (3), (4), (9) and

$$u = u_T \text{ in } \Omega \times \{T\} \tag{10}$$

hold.

IP2: Let $a_t = 0$. Given $\mu, m, \beta, a_{ij}, a_j, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, **find** a **and** u so that the relations (3), (4) and (10) hold.

IP3: Given $\mu, m, a_{ij}, a_j, a, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, **find** β **and** u so that the relations (3), (4) and (10) hold.

Define the resolvent kernel $\widehat{m}$ of the kernel m as the solution of the following Volterra integral equation:

$$\widehat{m}(t) - \int_0^t m(t-\tau)\widehat{m}(\tau)d\tau = m(t), \quad t \in (0, T). \tag{11}$$

Bringing the derivative with respect to t into the integral $\mu * u$ and applying the operator $I + \widehat{m}$ to the equation (3) and the boundary condition (4) in case II we transform the relations (3), (4) to the following form:

$$\beta(u_t + k * u_t) = Au + f \ \text{ in } Q, \quad u = u_0 \ \text{ in } \Omega \times \{0\}, \ \ B_1 u = g \ \text{ in } S, \tag{12}$$

where

$$k = \mu + \mu * \widehat{m} + \widehat{m}, \tag{13}$$

$$f = \chi - \beta\mu u_0 + \widehat{m} * (\chi - \beta\mu u_0), \tag{14}$$

$$B_1 = B, \ \ g = b \ \text{ in case I}, \tag{15}$$

$$B_1 u = \omega \cdot \nabla u, \ \ g = b + \widehat{m} * b \ \text{ in case II}. \tag{16}$$

Positivity principle

Theorem 1

Assume $k \in W_1^1(0,T)$, $\beta, a_{ij}, a_j \in C(\overline{\Omega})$, $a \in C(\overline{Q})$ *and*

$$k \geq 0, \quad k' \leq 0. \tag{17}$$

Let $u \in W_p^{2,1}(Q)$ *with some* $p \in (1,\infty)$ *solve the problem* (12) *and* $u_0 \geq 0$, $g \geq 0$, $f \geq 0$. *Then the following assertions are valid:*

(i) $u \geq 0$;

(ii) *if, in addition,* $\beta, a_{ij}, a_j \in C^l(\overline{\Omega})$, $a \in C^{l,\frac{l}{2}}(Q)$ *with some* $l \in (0,1)$ *and there exists an open subset* Q_f *of* Q *such that* $f > 0$ *in* Q_f, *then* $u(\cdot,T) > 0$ *in* Ω *in case* I *and* $u(\cdot,T) > 0$ *in* $\overline{\Omega}$ *in case* II.

Results for IP1

IP1 is in the class of pairs (z, u) of functions, whose second component u together with its derivatives u_t, u_{x_i}, $u_{x_i x_j}$ belongs to $L^p(0, T)$, $p > 1$, for any x, equivalent to the following inverse problem:

$$\begin{aligned} &\beta(u_t + k * u_t) = Au + zr + f_0 \ \text{ in } Q, \\ &u = u_0 \ \text{ in } \Omega \times \{0\}, \ \ B_1 u = g \ \text{ in } S, \end{aligned} \tag{18}$$

$$u \ = \ u_T \ \text{ in } \Omega \times \{T\}, \tag{19}$$

where B_1, g are given by (15), (16) and

$$r \ = \ \phi + \widehat{m} * \phi \,, \ \ f_0 = \chi_0 - \beta\mu u_0 + \widehat{m} * (\chi_0 - \beta\mu u_0). \tag{20}$$

Uniqueness theorem

Theorem 2

Let $k \in W_1^1(0,T)\,, \quad k \geq 0\,, \quad k' \leq 0$ *hold and* $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$ *with some* $l \in (0,1), p \in (1,\infty)$. *Moreover, let* $a_t \geq 0$ *in* Q, $r \in C^{l,\frac{l}{2}}(Q), r_t \in L^p(Q)$ *and*

$$r \geq 0\,, \;\; r_t + k * r_t - \theta r \geq 0 \quad \text{in } \; Q, \tag{21}$$

where $\theta = \sup\limits_{x\in\Omega} \dfrac{a(x,T)}{\beta(x)}$. *Finally, assume that*

$$\begin{aligned} &\text{for all } \; x \in \Omega \;\; \text{there exists an open subset } Q_x \text{ of } Q \text{ such that} \\ &\quad \exists t_x \in (0,T) \; : \; (x,t_x) \in Q_x \quad \text{and} \quad r_t + k * r_t - \theta r > 0 \;\; \text{in } Q_x. \end{aligned} \tag{22}$$

If $(z,u) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *solves* (18), (19) *and* $f_0, u_0, g, u_T = 0$ *then* $z = 0,\; u = 0$.

Existence and stability

Theorem 3

Let $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$ *and* $a_t \in L^p(Q)$ *with some* $l \in (0,1), p \in (1,\infty)$. *Moreover, let* $a_t \geq 0$, $r \in C^{l,\frac{l}{2}}(Q)$, $r_t \in L^p(Q)$ *and* $r \geq \delta$ *in* $\overline{\Omega} \times (T-\delta, T)$ *with some* $\delta \in (0, \frac{T}{2})$ *and* $r = 0$ *in* $\overline{\Omega} \times (0,\delta)$ *hold. In addition, let* $f_0 \in C^{l,\frac{l}{2}}(Q)$, $u_0 \in C^{2+l}(\Omega)$, $g \in C^{2+l-\vartheta, 1+\frac{l}{2}-\frac{\vartheta}{2}}(S)$, $u_T \in C^{2+l}(\Omega)$ *and the consistency conditions*

$$\text{(a)} \quad \begin{array}{l} u_0 = g\,, \quad \beta g_t = Au_0 + f_0 \quad \textit{in case } \mathrm{I}, \\ \omega \cdot \nabla u_0 = g \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{0\} \end{array} \tag{23}$$

$$\text{(b)} \quad \begin{array}{l} u_T = g \;\; \textit{in case } \mathrm{I}, \\ \omega \cdot \nabla u_T = g \quad \textit{in case } \mathrm{II} \quad \textit{in } \Gamma \times \{T\} \end{array} \tag{24}$$

be satisfied. Then the following assertions are valid.

Existence and stability

Theorem 3

(i) *(Fredholm-type result)*
If $k \in W^1_{\frac{2}{2-l}}(0,T), r \geq 0, r_t - \theta r \geq 0$ *in* Q *with* $\theta = \sup_{x\in\Omega} \frac{a(x,T)}{\beta(x)}$, *for all* $x \in \Omega$ *exists an open subset* $\tilde{Q}_x$ *of* Q *such that* $\exists \tilde{t}_x \in (0,T)$: $(x, \tilde{t}_x) \in \tilde{Q}_x$ *and* $r_t - \theta r > 0$ *in* $\tilde{Q}_x$ (25)

and the homogeneous inverse problem, i.e.

$$\beta(v_t^0 + k * v_t^0) = Av^0 + q^0 r \ \ in\ Q, \tag{26}$$

$$v^0 = 0 \ \ in\ \Omega\times\{0\}, \ \ B_1 v^0 = 0 \ \ in\ S, \ \ v^0 = 0 \ \ in\ \Omega\times\{T\} \tag{27}$$

has in $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *only the trivial solution* $q^0 = 0$, $v^0 = 0$, *then the inverse problem* (18), (19) *has a unique solution* (z,u) *in the space* $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$.

Existence and stability

Theorem 3

Moreover, the solution (z, u) satisfies the estimate

$$\|z\|_l + \|u\|_{2+l,1+\frac{l}{2}} \le \Lambda(\beta, a_{ij}, a_j, a, k, r) \times \left\{ \|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}} + \|u_T\|_{2+l} \right\} \qquad (28)$$

with some constant Λ depending on the quantities shown in brackets.

(ii) *(Full existence, uniqueness and stability result)* *If $k \in W^1_{\frac{2}{2-l}}(0,T)\,,\quad k \ge 0\,,\quad k' \le 0$ and r satisfies (21), (22), (25) then the inverse problem (18), (19) has a unique solution (z, u) in the space $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$. The solution satisfies the estimate (28).*

Making use of results proved for IP1, we have proved global uniqueness and local existence and stability of solutions to IP2 and IP3.

Non-smooth problems

Let (1) have the following form:

$$u_t + (\mu * u)_t = Au - m * Au + f + \nabla \cdot \phi + \varphi_t \quad \text{in } Q, \tag{29}$$

where f, φ are regular scalar functions and ϕ is a regular vector function, A is of the divergence type and has symmetric principal part, i.e. $(Av)(x) = \sum_{i,j=1}^{n} \left(a_{ij}(x) v_{x_j}\right)_{x_i} + a(x)v(x), \quad a_{ij} = a_{ji}.$

$$u = u_0 \quad \text{in } \Omega \times \{0\} \tag{30}$$

$$u = g \quad \text{in } \Gamma_1 \times (0, T), \tag{31}$$

$$-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = h + \nu \cdot \phi \quad \text{in } \Gamma_2 \times (0, T), \tag{32}$$

where the functions u_0, g, h are given and $\nu_A = \left(\sum_{j=1}^{n} a_{ij} \nu_j \Big|_{i=1,\dots,n} \right)$ is the co-normal vector to Γ and $\Gamma = \Gamma_1 \cup \Gamma_2, \mathrm{meas} \Gamma_1 \cap \Gamma 2 = 0.$

Inverse problems

Let us pose formal inverse problems. They use instant and integral data of the solution of (29) - (32).

IP4: Let the component f of the free term be of the form

$$f(x,t) = f_0(x,t) + \sum_{j=1}^{N} \gamma_j(t)\omega_j(x) \tag{33}$$

and $\mu = 0$, $\varphi = 0$. Given $m, a_{ij}, a, u_0, f_0, \phi, g, h, \gamma_j$, $j = 1, \ldots, N$, and functions $u_{T_i}(x)$, $x \in \Omega$, $i = 1, \ldots, N$ with $0 < T_1 < T_2 < \ldots < T_N \leq T$, **find** ω_j, $j = 1, \ldots, N$, such that the solution u of (29) - (32) satisfies the following instant additional conditions:

$$u = u_{T_i} \quad \text{in } \Omega \times \{T_i\}, \quad i = 1, 2, \ldots, N.$$

IP5: Let the component f of the free term be of the form (33) and $\mu = 0$, $\varphi = 0$. Given $m, a_{ij}, a, f_0, \phi, g, h, \gamma_j$, $j = 1, \dots, N$, and functions $v_i(x)$, $x \in \Omega$, $i = 1, \dots, N+1$, **find** ω_j, $j = 1, \dots, N$, **and** u_0 such that the solution u of (29) - (32) satisfies the following integral additional conditions:

$$\int_0^T \kappa_i(x,t)u(x,t)dt = v_i(x)\,,\ x \in \Omega\,, \quad i = 1, 2, \dots, N+1, \tag{34}$$

where κ_i, $i = 1, \dots, N+1$ are given weights.

IP6: Let $\mathrm{meas}\,\Gamma_2 > 0$. Given $a_{ij}, u_0, f, \phi, \varphi, g, h$ and functions $u_T(x)$, $x \in \Omega$, $v_i(t)$, $t \in (0, T)$, $i = 1, 2$, **find** a, m **and** μ such that the solution of (29) - (32) satisfies the following final and integral additional conditions:

$$u = u_T \quad \text{in } \Omega \times \{T\}, \tag{35}$$

$$\int_{\Gamma_2} \kappa_i(x,t)u(x,t)d\Gamma \;=\; v_i(t)\,,\ t \in (0, T), \ \ i = 1, 2, \tag{36}$$

where κ_i, $i = 1, 2$, are weights and $d\Gamma$ is the surface measure on Γ.

We introduce t-dependent cylinders

$$\Gamma_{1,T} = \Gamma_1 \times (0, T), \quad \Gamma_{2,T} = \Gamma_2 \times (0, T).$$

In the treatment of the weak direct problem we make use of the following functional spaces:

$$\begin{aligned}
\mathcal{U}(Q) &= C([0, T]; L^2(\Omega)) \cap L^2(0, T; W_2^1(\Omega)),\\
\mathcal{U}_0(Q) &= \Big\{\eta \in \mathcal{U}(Q) \ : \ \eta|_{\Gamma_{1,T}} = 0 \ \text{ in case } \Gamma_1 \neq \emptyset\Big\},\\
\mathcal{T}(Q) &= \Big\{\eta \in L^2(0, T; W_2^1(\Omega)) \ : \ \eta_t \in L^2(0, T; L^2(\Omega))\Big\},\\
\mathcal{T}_0(Q) &= \Big\{\eta \in \mathcal{T}(Q) \ : \ \eta|_{\Gamma_{1,T}} = 0 \ \text{ in case } \Gamma_1 \neq \emptyset\Big\}.
\end{aligned}$$

Let us collect other regularity assumptions on the data of the direct problem (29) - (32). They are

$$a_{ij} \in L^\infty(\Omega), \tag{37}$$

$$a \in L^{q_1}(\Omega), \ \ \text{where} \ \ q_1 = 1 \ \text{ if } \ n = 1, \ \ q_1 > \frac{n}{2} \ \text{ if } \ n \geq 2, \tag{38}$$

$$\mu \in L^2(0,T), \tag{39}$$

$$m \in L^1(0,T), \tag{40}$$

$$u_0 \in L^2(\Omega), \tag{41}$$

$$g \in \mathcal{T}(Q), \ \ h \in L^2(\Gamma_{2,T}), \tag{42}$$

$$\begin{gathered} f \in L^2(0,T; L^{q_2}(\Omega)), \ \ \text{where} \\ q_2 = 1 \ \text{ if } \ n = 1, \ \ q_2 \in (1, q_1) \ \text{ if } \ n = 2, \ \ q_2 = \tfrac{2n}{n+2} \ \text{ if } \ n \geq 3, \end{gathered} \tag{43}$$

$$\phi = (\phi_1, \ldots, \phi_n) \in (L^2(Q))^n, \tag{44}$$

$$\varphi \in \mathcal{U}(Q) \ \text{ and in case } \ \Gamma_1 \neq \emptyset \ \ \exists g_\varphi \in \mathcal{T}(Q) \ : \ \ \varphi = g_\varphi \ \text{ in } \Gamma_{1,T}. \tag{45}$$

$$\int_\Omega \Big[(u+\mu * u-\varphi)(x,T)\eta(x,T)-(u_0(x)-\varphi(x,0))\eta(x,0)\Big]dx$$

$$-\iint\limits_Q (u+\mu * u-\varphi)\eta_t\,dxdt \quad (46)$$

$$+\iint\limits_Q \Big[\sum_{i,j=1}^{n} a_{ij}(u_{x_j}-m * u_{x_j})\eta_{x_i}-a(u-m * u)\eta\Big]dxdt$$

$$+\iint\limits_{\Gamma_{2,T}} h\eta\,d\Gamma dt-\iint\limits_Q (f\eta-\phi\cdot\nabla\eta)\,dxdt=0.$$

We call a *weak solution* of the problem (29) - (32) a function that belongs to $\mathcal{U}(Q)$, satisfies the relation (46) for any $\eta \in \mathcal{T}_0(Q)$ and, in case $\Gamma_1 \neq \emptyset$, fulfills the boundary condition (31).

Theorem 5

Let (37) - (45) *hold. Then the problem* (29) - (32) *has a unique weak solution* $u \in \mathcal{U}(Q)$*.*
The function $u \in \mathcal{U}(Q)$ *satisfies the relation* (46) *for any* $\eta \in \mathcal{T}_0(Q)$ *if and only if it satisfies the following relation*

$$\int_\Omega (u + \mu * u - \varphi) * \eta \, dx - \int_\Omega \int_0^t (u_0(x) - \varphi(x,0))\eta(x,\tau) d\tau dx \tag{47}$$

$$+ \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta \Big] dx$$

$$+ \int_{\Gamma_2} 1 * h * \eta \, d\Gamma - \int_\Omega 1 * \Big(f * \eta - \sum_{i=1}^n \phi_i * \eta_{x_i} \Big) dx = 0, \quad t \in [0, T],$$

for any $\eta \in \mathcal{U}_0(Q)$*.*

Quasi-solutions

(1) Firstly, let us consider IP4. We look for the vector of unknowns $\omega = (\omega_1, \ldots, \omega_N)$ in the space $\mathcal{Z}_1 = (L^2(\Omega))^N$. Assume that $\mu = 0$, $\varphi = 0$, (37), (38), (40) - (42), (44) hold, f_0 satisfies (43) and $\gamma_j \in L^2(0,T)$, $j = 1, \ldots, N$. Then, by Theorem 5, the problem (29) - (32) with f of the form (33) has a unique weak solution $u \in \mathcal{U}(Q)$ for any $\omega \in \mathcal{Z}_1$. We denote this ω-dependent solution by $u(x,t;\omega)$.

Let $M \subseteq \mathcal{Z}_1$. Assume $u_{T_i} \in L^2(\Omega)$, $i = 1, \ldots, N$. The quasi-solution of IP4 in the set M is an element $\omega^* \in \arg\min_{\omega \in M} J_1(\omega)$, where J_1 is the following cost functional:

$$J_1(\omega) = \sum_{i=1}^{N} \|u(x, T_i; \omega) - u_{T_i}(x)\|^2_{L^2(\Omega)}.$$

(2) In IP5 we search for vectors $z = (\omega, u_0) \in \mathcal{Z}_2 = (L^2(\Omega))^{N+1}$. Assume that $\mu = 0$, $\varphi = 0$, (37), (38), (40), (42), (44) hold, f_0 satisfies (43) and $\gamma_j \in L^2(0,T)$, $j = 1, \ldots, N$. Then the problem (29) - (32) with f of the form (33) has a unique weak solution $u = u(x,t;z) \in \mathcal{U}(Q)$ for any $z \in \mathcal{Z}_2$.

Further, let $M \subseteq \mathcal{Z}_2$ and assume that $\kappa_i \in L^\infty(Q)$, $v_i \in L^2(\Omega)$, $i = 1, \ldots, N+1$. The quasi-solution of IP5 in the set M is $z^* \in \arg\min_{z \in M} J_2(z)$, where J_2 is the cost functional

$$J_2(z) = \sum_{i=1}^{N+1} \left\| \int_0^T \kappa_i(\cdot,t) u(\cdot,t;z) dt - v_i \right\|_{L^2(\Omega)}^2 .$$

(3) In IP6 we look for the vector $z = (a, m, \mu) \in \mathcal{Z}_3 = L^2(\Omega) \times (L^2(0, T))^2$. Assume that $n \in \{1; 2; 3\}$. This guarantees that any $a \in L^2(\Omega)$ satisfies (38). Moreover, assume that (37), (41) - (45) hold, where $q_2 \in (1, 2)$ in (43) in case $n = 2$. Under such assumptions the problem (29) - (32) has a unique weak solution $u = u(x, t; z) \in \mathcal{U}(Q)$ for any $z \in \mathcal{Z}_3$.

Let $M \subseteq \mathcal{Z}_3$ and assume that $u_T \in L^2(\Omega)$, $\kappa_i \in L^\infty(\Gamma_{2,T})$, $v_i \in L^2(0, T)$, $i = 1, 2$. The quasi-solution of IP6 in the set M is $z^* \in \arg\min_{z \in M} J_3(z)$, where J_3 is the cost functional

$$J_3(z) = \|u(\cdot, T; z) - u_T\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left\| \int_{\Gamma_2} \kappa_i(x, \cdot) u(x, \cdot; z) d\Gamma - v_i \right\|^2_{L^2(0,T)}.$$

Derivative of J_1

Theorem 6

Let the assumptions listed in (1) *be satisfied. Then the functional J_1 is Fréchet differentiable in $\mathcal{Z}_1$ and $J_1'(\omega)\Delta\omega = \langle \varrho_1, \Delta\omega \rangle_{\mathcal{Z}_1}$, where the ω-dependent vector $\varrho_1 = \varrho_1(x;\omega)$ consists of the components*

$$\varrho_{1,j}(x;\omega) = \sum_{i=1}^{N} \int_0^{T_i} \gamma_j(t)\psi_i(x, T_i - t;\omega)dt, \; j = 1, \ldots, N, \tag{48}$$

$\psi_i = \psi_i(x,t;\omega) \in \mathcal{U}(Q)$, $i = 1, \ldots, N$, are the unique ω-dependent weak solutions of the following (adjoint) problems:

Theorem 6

$$\psi_{i,t} = A\psi_i - m * A\psi_i \quad in\ Q_{T_i}, \tag{49}$$

$$\psi_i = 2[u(x, T_i; \omega) - u_{T_i}(x)] \quad in\ \Omega \times \{0\}, \tag{50}$$

$$\psi_i = 0 \quad in\ \Gamma_{1,T_i}, \tag{51}$$

$$-\nu_A \cdot \nabla\psi_i + m * \nu_A \cdot \nabla\psi_i = 0 \quad in\ \Gamma_{2,T_i} \tag{52}$$

and $\langle \varrho_1, \omega \rangle_{\mathcal{Z}_1} = \sum_{j=1}^{N} \langle \varrho_{1,j}, \omega_j \rangle_{L^2(\Omega)}$ *is the inner product of* ϱ_1 *and* ω *in the space* $\mathcal{Z}_1$.

Analogously we can formulate results for Frechet derivative of J_2 and J_3.

Existence of quasi-solutions

Theorem 7

Let the assumptions listed in (1) *be satisfied and* $M \subset \mathcal{Z}_1$ *be compact. Then* IP4 *has a quasi-solution in* M*. Similar assertions are valid for* IP5 *and* IP6*, too.*

Theorem 8

Let the assumptions listed in (1) *be satisfied and* $M \subset \mathcal{Z}_1$ *be bounded, closed and convex. Then* IP4 *has a quasi-solution in* M*. The set of quasi-solutions is closed and convex. Similar assertion is valid for* IP5*, too.*

Theorem 9

Let the assumptions listed in (3) *be satisfied. Assume that* $n = 1$, $\Omega = (c, d)$, $\varphi = 0$, $g(\cdot, 0) = 0$ *and* M *be bounded, closed and convex. Then* IP6 *has a quasi-solution in* M.

Thank you!